Some correct and selfadjoint problems with differential cubic operators

Th. Lokkas *

March 2, 2009

Abstract

In this paper are given some applications concerning correct and selfadjoint boundary problems with differential cubic operators. Also the solutions of these problems are obtained.

1 Introduction

Correct and selfadjoint boundary problems with cubic operators have been studied by I.N. Parasidis and P.C. Tsekrekos in the paper entitled "Correct and selfadjoint boundary problems with cubic operators " [2] which is going to be presented in the Conference "Computer Algebra" in St. Petersburg, Russia 2009. In this paper are given applications of the above theory and are studied specific boundary problems, with integro-differential equations, which reduced to the type

$$B_3x = \widehat{A}^3x - Y\langle\widehat{A}x, F^t\rangle_{H^m} - S\langle\widehat{A}^2x, F^t\rangle_{H^m} - G\langle\widehat{A}^3x, F^t\rangle_{H^m} = f,\ D(B_3) = D(\widehat{A}^3), \tag{1.1}$$

where $\widehat{A}$ is one well known correct selfadjoint operator, the vectors $Y \in \mathbb{H}^m$, $S \in D(\widehat{A})^m$, $G \in D(\widehat{A}^2)^m$, $F \in D(\widehat{A}^3)^m$ and S, Y satisfy (3.3).
If an operator B_1 is not cubic i.e. S, Y is not satisfy (3.3), then the correctness and selfadjointness of the problems $B_1x = f$ can be proved by the method developed in [1]. But if B_1 is cubic i.e. $B_1 = B_3$, the proof of the correctness and selfadjointness is much simpler.
The paper is organized as follows. In Section 2 we recall some basic terminology and notation about operators. In Section 3 we recall the theory of correct and

*Technological Educational Institution of Larissa, Greece

selfadjoint boundary problems with cubic operators, give the remark 3.3, we prove also the proposition 3.4 and, finally, consider some applications of this theory.

2 Terminology and notation

By $\langle x, f\rangle_H$ is denoted the inner product of elements x, f of a complex Hilbert space $\mathbb{H}$. For operators $A : \mathbb{H} \to \mathbb{H}$ we write $D(A)$ and $R(A)$ for the domain and the range of A respectively. An operator $\widehat{A}$ is called *correct* if $R(\widehat{A}) = \mathbb{H}$ and the inverse $\widehat{A}^{-1}$ exists and is continuous on $\mathbb{H}$. Let A be an operator with domain dense in $\mathbb{H}$. The *adjoint* operator $A^* : \mathbb{H} \longrightarrow \mathbb{H}$ of A with domain $D(A^*)$ is defined by the equation $\langle Ax, y\rangle_H = \langle x, A^*y\rangle_H$ for every $x \in D(A)$ and every $y \in D(A^*)$. The domain $D(A^*)$ of A^* consists of all $y \in \mathbb{H}$ for which the functional $x \longmapsto \langle Ax, y\rangle_H$ is continuous on $D(A)$. An operator A is called *selfadjoint* if $A = A^*$. An operator D is called *cubic* if there exists an operator B such that $D = B^3$. Let $F_i \in \mathbb{H}, i = 1, \ldots, m$. Then $F = (F_1, \ldots, F_m)$ and $AF = (AF_1, \ldots, AF_m)$ are vectors of $\mathbb{H}^m$ and $\mathcal{F} = (\widehat{A}^{-2}F, \widehat{A}^{-1}F, F) = (\widehat{A}^{-2}F_1, \ldots, \widehat{A}^{-2}F_m, \widehat{A}^{-1}F_1, \ldots, \widehat{A}^{-1}F_m, F_1, \ldots, F_m)$ is a vector of $\mathbb{H}^{3m}$. We also write F^t and $\langle Ax, F^t\rangle_{H^m}$ for the column vectors $col(F_1, \ldots, F_m)$ and
$col(\langle Ax, F_1\rangle_H, \ldots, \langle Ax, F_m\rangle_H)$ respectively and denote by I_m the identity $m\times m$ matrix. By $\widehat{A}^{-3}$ is denoted the operator $(\widehat{A}^{-1})^3$, by N^t the transpose matrix of N, by $\langle AF^t, F\rangle_{H^m}$ the $m \times m$ matrix whose i, j-th entry is the inner product $\langle AF_i, F_j\rangle_H$ and by $\langle AF^t, F\rangle_{H^m}$ the $m \times m$ matrix whose i, j-th entry is the inner product $\langle AF_i, F_j\rangle_H$.

3 Some correct and selfadjoint problems with differential cubic operators

We shall make use of the following [2, Lemma 3.3, Theorem 3.4]

Lemma 3.1. *Let the operators $B, B_3 : \mathbb{H} \to \mathbb{H}$ be defined by*

$$Bx = \widehat{A}x - G\langle \widehat{A}x, F^t\rangle_{H^m} = f, \qquad D(B) = D(\widehat{A}), \tag{3.1}$$

$$B_3x = \widehat{A}^3x - Y\langle \widehat{A}x, F^t\rangle_{H^m} - S\langle \widehat{A}^2x, F^t\rangle_{H^m} - G\langle \widehat{A}^3x, F^t\rangle_{H^m} = f,\ D(B_3) = D(\widehat{A}^3), \tag{3.2}$$

where $\widehat{A}$ *is a selfadjoint operator on* $\mathbb{H}$, *a vector* $G \in D(\widehat{A}^2)^m$,

$$S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m}, \qquad Y = \widehat{A}S - G\overline{\langle F^t, \widehat{A}S\rangle}_{H^m} \tag{3.3}$$

and the components of the vector $F = (F_1, \ldots, F_m)$ *belong to* $D(\widehat{A}^3)$. *Then* $B_3 = B^3$, *i.e.* B_3 *is a cubic operator.*

Theorem 3.2. *Let the operators* $\widehat{A}, B_3 : \mathbb{H} \to \mathbb{H}$ *and vectors* G, S, Y *be defined as in lemma 3.1. We suppose also that* $\widehat{A}$ *is a correct operator,* $G = (\widehat{A}F)C$, *where* C *is a* $m \times m$ *matrix with rank* $C = n \leq m$ *and the components of vector* $\mathcal{F} = (\widehat{A}^{-2}F, \widehat{A}^{-1}F, F)$ $\big($*resp.* $\widehat{A}^2\mathcal{F} = (F, \widehat{A}F, \widehat{A}^2F)\big)$ *are linearly independent elements of* $D(\widehat{A}^3)$ $\big($*resp.* $D(\widehat{A})\big)$. *Then:*
(i) B_3 *is selfadjoint if and only if* C *is Hermitian,*
(ii) $\dim R(B_3 - \widehat{A}^3) = 3n \, (n \leq m)$,
(iii) B_3 *is a correct operator if and only if holds true*

$$\det L = \det \left[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C \right] \neq 0. \tag{3.4}$$

(iv) The unique solution, for every $f \in \mathbb{H}$, *of the problem (3.2) is given by*

$$\begin{aligned} x =& B_3^{-1}f = \widehat{A}^{-3}f + \big[\widehat{A}^{-2}F + (\widehat{A}^{-1}F)CL^{-1}\overline{\langle F^t, F\rangle}_{H^m} + FCL^{-1}\big(\overline{\langle \widehat{A}^{-1}F^t, F\rangle}_{H^m} \\ &+ \overline{\langle F^t, F\rangle}_{H^m} CL^{-1}\overline{\langle F^t, F\rangle}_{H^m}\big)\big] CL^{-1}\langle f, F^t\rangle_{H^m} + \big[\widehat{A}^{-1}F \\ &+ FCL^{-1}\overline{\langle F^t, F\rangle}_{H^m}\big] CL^{-1}\langle f, \widehat{A}^{-1}F^t\rangle_{H^m} + FCL^{-1}\langle f, \widehat{A}^{-2}F^t\rangle_{H^m}. \end{aligned} \tag{3.5}$$

Remark 3.3. In applications we encounter operators B_1 of the form

$$\begin{gathered} B_1 u = \widehat{A}^3 u - V_{1m}\langle u, J_1^t\rangle_{H^m} - V_{2m}\langle u, J_2^t\rangle_{H^m} - V_{3m}\langle u, J_3^t\rangle_{H^m} = f, \\ D(B_1) = D(\widehat{A}^3), \end{gathered} \tag{3.6}$$

where the vectors J_i, $V_{im} \in \mathbb{H}^m$, $i = 1, 2, 3$. Then we are interested to know if the operator B_1 is a B_3 operator defined by (3.2) and so to apply the theorem 3.2. To this end we work as follows:
1. we show that the operator $\widehat{A}$ in (3.6) is correct and selfadjoint,
2. we find a vector $F \in D(\widehat{A}^3)^m$ and $m \times m$ matrices M_i, $i = 1, 2, 3$ with constant elements such that $\langle u, J_1^t\rangle_{H^m} = M_1\langle \widehat{A}u, F^t\rangle_{H^m}$,
$\langle u, J_2^t\rangle_{H^m} = M_2\langle \widehat{A}^2u, F^t\rangle_{H^m}$ and $\langle u, J_3^t\rangle_{H^m} = M_3\langle \widehat{A}^3u, F^t\rangle_{H^m}$,
3. we find vectors $Y = V_{1m}M_1$, $S = V_{2m}M_2 \in \mathbb{H}^m$ and $G = V_{3m}M_3 \in D(\widehat{A})^m$ such that $Y = \widehat{A}S - G\overline{\langle F^t, \widehat{A}S\rangle}_{H^m}$ and $S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m}$.
If one of these steps fails, then B_1 is not identified as an B_3 operator and so the theory can not be applied.

Bellow $H^i(0,1)$ denote the Sobolev spaces of all complex functions of $L_2(0,1)$ which have generalized derivatives up to i -th order respectively Lebesque integrable, $i = 1, 2, 3, 4$.

We recall [1, p.780] that the operator $\widehat{A} : L_2(0,1) \to L_2(0,1)$ defined by

$$\widehat{A}u = iu' = f, \quad D(\widehat{A}) = \{u(t) \in H^1(0,1) : u(0) + u(1) = 0\} \tag{3.7}$$

is correct and selfadjoint and the unique solution u of the problem (3.7) is given by the formula

$$\widehat{A}^{-1}f = \frac{i}{2}\int_0^1 f(x)dx - i\int_0^t f(x)dx \quad \text{for all } f \in H. \tag{3.8}$$

Then it follows easily [1, p.781] that the operator $\widehat{A}^2$ defined by

$$\widehat{A}^2 u = -u'' = f, \quad D(\widehat{A}^2) = \{u \in H^2(0,1) : u(0) + u(1) = 0,\ u'(0) + u'(1) = 0\} \tag{3.9}$$

is correct and selfadjoint and for every $f \in L_2(0,1)$ the unique solution u of the problem (3.9) is given by the formula

$$u = \widehat{A}^{-2}f = -\int_0^t (t-x)f(x)dx + \frac{1}{4}\int_0^1 (2t - 2x + 1)f(x)dx. \tag{3.10}$$

Proposition 3.4. *The operator $\widehat{\mathcal{A}} : L_2(0,1) \to L_2(0,1)$ which corresponds to the problem*

$$\widehat{\mathcal{A}}u = -iu''' = f, \tag{3.11}$$
$$D(\widehat{\mathcal{A}}) = \{u \in H^3(0,1) : u(0) + u(1) = 0,\ u'(0) + u'(1) = 0,\ u''(0) + u''(1) = 0\}$$

is correct and selfadjoint and for every $f \in L_2(0,1)$ the unique solution u of the problem (3.11) is given by the formula

$$\begin{aligned} u(t) = \widehat{\mathcal{A}}^{-1}f = &\frac{i}{2}\int_0^t (t-x)^2 f(x)dx - \frac{i}{4}(t^2 - t)\int_0^1 f(x)dx \\ &-\frac{i}{4}\int_0^1 [2t - (2t+1)x + x^2]f(x)dx. \end{aligned} \tag{3.12}$$

Proof. It is evident that $\widehat{\mathcal{A}} = \widehat{A}^3$, where $\widehat{A}$ is defined by (3.7). Correctness and selfadjointness of $\widehat{A}$ implies correctness and selfadjointness of $\widehat{A}^3 = \widehat{\mathcal{A}}$. Now we will prove the formula (3.12). Let $g(x) = \widehat{A}^{-1}f(x)$. Then by (3.10), (3.8) and Fubini's theorem we have

$$\widehat{\mathcal{A}}^{-1}f(t)=\widehat{A}^{-3}f(t)=\widehat{A}^{-2}\big(\widehat{A}^{-1}f(t)\big)=\widehat{A}^{-2}g(x)=-\int_0^t(t-x)g(x)dx$$

$$+\frac{1}{4}\int_0^1(2t-2x+1)g(x)dx=-\int_0^t(t-x)\Big[\frac{i}{2}\int_0^1 f(y)dy-i\int_0^x f(y)dy\Big]dx$$

$$+\frac{1}{4}\int_0^1(2t-2x+1)\Big[\frac{i}{2}\int_0^1 f(y)dy-i\int_0^x f(y)dy\Big]dx=-\frac{i}{2}\int_0^t(t-x)dx\int_0^1 f(y)dy$$

$$+i\int_0^t(t-x)dx\int_0^x f(y)dy+\frac{i}{8}\int_0^1(2t-2x+1)dx\int_0^1 f(y)dy$$

$$-\frac{i}{4}\int_0^1(2t-2x+1)dx\int_0^x f(y)dy=-\frac{it^2}{4}\int_0^1 f(y)dy+\frac{i}{2}\int_0^t(t-y)^2f(y)dy$$

$$+\frac{it}{4}\int_0^1 f(y)dy-\frac{i}{4}\int_0^1[2t-(2t+1)y+y^2]f(y)dy \quad \text{and finally we obtain}$$

$$\widehat{A}^{-3}f=\frac{i}{2}\int_0^t(t-x)^2f(x)dx-\frac{i}{4}(t^2-t)\int_0^1 f(x)dx \tag{3.13}$$

$$-\frac{i}{4}\int_0^1\big[2t-(2t+1)x+x^2\big]f(x)dx.$$

which, since $\widehat{\mathcal{A}}^{-1}=\widehat{A}^{-3}$, gives (3.12). □

Example 3.5. The operator $B_1 : L_2(0,1)\to L_2(0,1)$ which corresponds to the problem

$$\begin{aligned}B_1u=&-iu'''+120c_1[2c_1^2(t^2-t)+ic_1(2t-1)-1]\int_0^1(x^2-x)u(x)dx\\&+5c_1[2t-1-2ic_1(t^2-t)]\int_0^1 u''(x)(4x^3-6x^2+1)dx\\&+5c_1(t^2-t)\int_0^1 u'''(x)(4x^3-6x^2+1)dx=f(t),\quad D(B_1)=D(\widehat{\mathcal{A}})\end{aligned} \tag{3.14}$$

is correct and selfadjoint iff c_1 is a real nonzero constant. The unique solution of (3.14), for each $f\in L_2(0,1)$, is given by the formula

$$\begin{aligned}u(t)=&\widehat{A}^{-3}f(t)+\frac{5c_1}{12}\Big[\frac{1}{10}(2t^5-5t^4+5t^2-1)-\frac{17ic_1}{84}(t^4-2t^3+t)+\\&+\frac{289c_1^2}{7056}(4t^3-6t^2+1)\Big]\int_0^1(4x^3-6x^2+1)f(x)dx-\frac{5c_1}{12}\Big[t^4-2t^3+t\\&-\frac{17ic_1}{84}(4t^3-6t^2+1)\Big]\int_0^1(x^4-2x^3+x)f(x)dx+\frac{c_1}{24}(4t^3-6t^2\\&+1)\int_0^1(2x^5-5x^4+5x^2-1)f(x)dx,\end{aligned} \tag{3.15}$$

where $\widehat{A}^{-3}f(t)$ is defined by (3.13).

Proof. We refer to theorem 3.2. If we compare equation (3.14) with equation (3.2) it is natural to take $\widehat{A}^3 u = -iu'''$ with $D(\widehat{A}^3) = D(B_1)$, $m = 1$, $F = 4t^3 - 6t^2 + 1$. Then we can take $\widehat{A}$ to be defined by (3.7), $\widehat{A}^2$ by (3.9). It evident that $F \in D(\widehat{A}^3)$, $\widehat{A}F = 12i(t^2 - t)$, $\widehat{A}^2 F = -12(2t-1)$, and that $\langle \widehat{A}u, F\rangle_H = \int_0^1 iu'(x)(4x^3 - 6x^2 + 1)dx$, $\langle \widehat{A}^2 u, F\rangle_H = -\int_0^1 u''(x)(4x^3 - 6x^2 + 1)dx$, $\langle \widehat{A}^3 u, F\rangle_H = \int_0^1 [-iu'''(x)](4x^3 - 6x^2 + 1)dx$. By integrating by parts we have $\langle \widehat{A}u, F\rangle_H = -12i\int_0^1 (x^2 - x)u(x)dx$. Then $\int_0^1 (x^2 - x)u(x)dx = \frac{i}{12}\langle \widehat{A}u, F\rangle_H$, $\int_0^1 u''(x)(4x^3 - 6x^2 + 1)dx = -\langle \widehat{A}^2 u, F\rangle_H$, $\int_0^1 u'''(x)(4x^3 - 6x^2 + 1)dx = i\langle \widehat{A}^3 u, F\rangle_H$. Replacing these elements in (3.14) we get:

$$\begin{aligned} B_1 u = \widehat{A}^3 u + 10ic_1[2c_1^2(t^2 - t) + ic_1(2t-1) - 1]\langle \widehat{A}u, F\rangle_H - 5c_1[2t - 1 \\ -2ic_1(t^2 - t)]\langle \widehat{A}^2 u, F\rangle_H + 5ic_1(t^2 - t)\langle \widehat{A}^3 u, F\rangle_H = f(t). \end{aligned} \tag{3.16}$$

By comparing again (3.16) with (3.2)) we take $Y = -10ic_1[2c_1^2(t^2 - t) + ic_1(2t - 1) - 1]$, $S = 5c_1[2t - 1 - 2ic_1(t^2 - t)]$ and $G = -5ic_1(t^2 - t)$. It evident that $G \in D(\widehat{A}^2)$ and $F, \widehat{A}F, \widehat{A}^2 F$ are linearly independent elements of $D(\widehat{A})$. By simple calculations we find $\widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} = 5c_1(2t-1) - 10ic_1^2(t^2 - t) = S$ and $\widehat{A}S - G\overline{\langle F^t, \widehat{A}S\rangle}_{H^m} = 10c_1[i + c_1(2t-1) - 2ic_1^2(t^2 - t)] = Y$. The last two equalities, by lemma 3.1, show that the operator B_1 is cubic, i.e. $B_1 = B_3$. From $G = (\widehat{A}F)C$ it follows $-5ic_1(t^2 - t) = 12i(t^2 - t)C$. This equation implies that $C = -5c_1/12$. We find $\langle F^t, F\rangle_H = 17/35$, $\langle \widehat{A}F^t, F\rangle_H = 0$. By theorem 3.2 the operator B_1 is correct and selfadjoint iff c_1 is a real number and $\det L = \det[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C] = 1 - 0 = 1 \neq 0$. Hence $L^{-1} = 1$. So B_1 is correct and selfadjoint if and only if c_1 is a real nonzero constant. If we substitute in (3.8) and (3.10) $f = F = 4t^3 - 6t^2 + 1$, we receive $\widehat{A}^{-1}F = -i(t^4 - 2t^3 + t)$ and $\widehat{A}^{-2}F = -\frac{1}{10}(2t^5 - 5t^4 + 5t^2 - 1)$. Then $\langle f, \widehat{A}^{-1}F\rangle_H = i\int_0^1 (x^4 - 2x^3 + x)f(x)dx$, $\langle f, \widehat{A}^{-2}F\rangle_H = -\frac{1}{10}\int_0^1 (2x^5 - 5x^4 + 5x^2 - 1)f(x)dx$ and $\langle \widehat{A}^{-1}F, F\rangle_H = 0$. From this and (3.5), (3.10) we get the solution (3.15) of the problem (3.14). □

Example 3.6. The operator $B_1 : L_2(0,1) \to L_2(0,1)$ which corresponds to the problem

$$\begin{aligned} B_1 u = &- iu''' + c_1\pi^2[(1 + 25c_1)\cos \pi t + (27 + 75c_1^2)\cos 3\pi t + 5i(\sin \pi t \\ &+9\sin 3\pi t)]\int_0^1 u'(x)(\sin \pi x + \sin 3\pi x)dx - c_1\pi^2[\sin \pi t + 9\sin 3\pi t \\ &-5ic_1(\cos \pi t + 3\cos 3\pi t)]\int_0^1 u'(x)(\cos \pi x + 3\cos 3\pi x)dx - c_1(\cos \pi t \\ &+3\cos 3\pi t)\int_0^1 u'''(x)(\sin \pi x + \sin 3\pi x)dx = f(t), \quad D(B_1) = D(\widehat{\mathcal{A}}) \end{aligned} \tag{3.17}$$

is correct and selfadjoint iff c_1 is a real nonzero constant.
The unique solution of (3.17), for each $f \in L_2(0,1)$, is given by the formula

$$
\begin{aligned}
u(t) =& \widehat{A}^{-3} f(t) + \frac{c_1}{9\pi^3}\Big\{[\sin 3\pi t + 9\sin\pi t + 3ic_1(3\cos\pi t + \cos 3\pi t) + 9c_1^2(\sin\pi t \\
&+ \sin 3\pi t)]\int_0^1 (\sin\pi x + \sin 3\pi x) f(x)dx + [3\cos\pi t + \cos 3\pi t - 3ic_1(\sin\pi t \\
&+ \sin 3\pi t)]\int_0^1 (3\cos\pi x + \cos 3\pi x) f(x)dx + (\sin\pi t \\
&+ \sin 3\pi t)\int_0^1 (9\sin\pi x + \sin 3\pi x) f(x)dx, \qquad (3.18)
\end{aligned}
$$

where $\widehat{A}^{-3} f(t)$ is defined by (3.13).

Proof. We refer to theorem 3.2. If we compare equation (3.14) with equation (3.2) it is natural to take $\widehat{A}^3 u = -iu'''$ with $D(\widehat{A}^3) = D(B_1)$, $m = 1$, $F = \sin\pi t + \sin 3\pi t$. Then we can take $\widehat{A}$ to be defined by (3.7), $\widehat{A}^2$ by (3.9). It evident that $F \in D(\widehat{A}^3)$, $\widehat{A}F = i\pi(\cos\pi t + 3\cos 3\pi t)$, $\widehat{A}^2 F = \pi^2(\sin\pi t + 9\sin 3\pi t)$ and that $\langle \widehat{A}u, F\rangle_H = \int_0^1 iu'(x)(\sin\pi x + \sin 3\pi x)dx$, $\langle \widehat{A}^2 u, F\rangle_H = -\int_0^1 u''(x)(\sin\pi x + \sin 3\pi x)dx = \pi\int_0^1 u'(x)(\cos\pi x + 3\cos 3\pi x)dx$, $\langle \widehat{A}^3 u, F\rangle_H = \int_0^1 [-iu'''(x)](\sin\pi x + \sin 3\pi x)dx$. Then $\int_0^1 u'(x)(\sin\pi x + \sin 3\pi x)dx = -i\langle \widehat{A}u, F\rangle_H$, $\int_0^1 u'(x)(\cos\pi x + 3\cos 3\pi x)dx = \frac{1}{\pi}\langle \widehat{A}^2 u, F\rangle_H$, $\int_0^1 u'''(x)(\sin\pi x + \sin 3\pi x)dx = i\langle \widehat{A}^3 u, F\rangle_H$. Replacing these elements in (3.17) we get:

$$
\begin{aligned}
B_1 u =& \widehat{A}^3 u - ic_1\pi^2[(1 + 25c_1)\cos\pi t + (27 + 75c_1^2)\cos 3\pi t + 5i(\sin\pi t \\
&+ 9\sin 3\pi t)]\langle \widehat{A}u, F\rangle_H - c_1\pi[\sin\pi t + 9\sin 3\pi t - 5ic_1(\cos\pi t \qquad (3.19) \\
&+ 3\cos 3\pi t)]\langle \widehat{A}^2 u, F\rangle_H - ic_1(\cos\pi t + 3\cos 3\pi t)\langle \widehat{A}^3 u, F\rangle_H = f(t).
\end{aligned}
$$

By comparing again (3.16) with (3.2)) we take $Y = ic_1\pi^2[(1+25c_1)\cos\pi t + (27 + 75c_1^2)\cos 3\pi t + 5i(\sin\pi t + 9\sin 3\pi t)]$, $S = c_1\pi[\sin\pi t + 9\sin 3\pi t - 5ic_1(\cos\pi t + 3\cos 3\pi t)]$ and $G = ic_1(\cos\pi t + 3\cos 3\pi t)$. It evident that $G \in D(\widehat{A}^2)$. The vectors $F, \widehat{A}F, \widehat{A}^2 F$ are linearly independent elements of $D(\widehat{A})$, since the corresponding determinant of the Gramm matrix is nonzero. By simple calculations we find $\widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} = c_1\pi[\sin\pi t + 9\sin 3\pi t - 5ic_1(\cos\pi t + 3\cos 3\pi t)] = S$ and $\widehat{A}S - G\overline{\langle F^t, \widehat{A}S\rangle}_{H^m} = ic_1\pi^2[\cos\pi t + 27\cos 3\pi t + 5ic_1(\sin\pi t + 9\sin 3\pi t)] - ic_1(\cos\pi t + 3\cos 3\pi t)(-25\pi^2 c_1^2) = Y$. The last two equalities, by lemma 3.1, show that the operator B_1 is cubic, i.e. $B_1 = B_3$. From $G = (\widehat{A}F)C$ it follows $ic_1(\cos\pi t + 3\cos 3\pi t) = i\pi(\cos\pi t + 3\cos 3\pi t)C$. This equation implies that $C = c_1/\pi$. We find $\langle F^t, F\rangle_H = 1$, $\langle \widehat{A}F^t, F\rangle_H = 0$. By theorem 3.2 the operator B_1 is correct and selfadjoint iff c_1 is a real number and $\det L = \det[I_m -$

$\overline{\langle \widehat{A}F^t, F\rangle_{H^m}}C] = 1-0 = 1 \neq 0$. Hence $L^{-1} = 1$. So B_1 is correct and selfadjoint if and only if c_1 is a real nonzero constant. If we substitute in (3.8) and (3.10) $f = F = \sin \pi t + \sin 3\pi t$, we receive $\widehat{A}^{-1}F = \frac{i}{3\pi}(3\cos \pi t + \cos 3\pi t)$ and $\widehat{A}^{-2}F = \frac{1}{9\pi^2}(9\sin \pi t + \sin 3\pi t)$. Then $\langle f, \widehat{A}^{-1}F\rangle_H = -\frac{i}{3\pi}\int_0^1 (3\cos \pi x + \cos 3\pi x) f(x)dx$, $\langle f, \widehat{A}^{-2}F\rangle_H = \frac{1}{9\pi^2}\int_0^1 (9\sin \pi x + \sin 3\pi x) f(x)dx$ and $\langle \widehat{A}^{-1}F, F\rangle_H = 0$. From this and (3.5), (3.10) we get the solution (3.18) of the problem (3.17). □

References

[1] I.N. Parasidis and P.C. Tsekrekos, *Correct selfadjoint and positive extensions of nondensely defined symmetric operators*,Abstract and Applied Analysis, Hindawi Publishing Corporation,Volue 2005:7,(2005)767-790.

[2] I.N. Parasidis and P.C. Tsekrekos, *Correct and selfadjoint problems with cubic operators* submitted to the Conference "Computer Algebra" in St. Petersburg, Russia 2009.