Some correct and selfadjoint problems with differential biquadratic operators.

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March 8, 2010

Abstract

Some applications concerning correct and selfadjoint boundary problems with differential biquadratic operators are given in this paper. The solutions of these problems are also obtained.

1 Introduction

Correct and selfadjoint boundary problems with biquadratic operators have been studied by I.N. Parasidis and P.C. Tsekrekos in the paper entitled "Correct and selfadjoint boundary problems with biquadratic operators" [**?**] which is going to be presented in the Conference "Computer Algebra" in St. Petersburg, Russia 2010. In this paper applications of the above theory are presented, while specific boundary problems, with integro-differential equations are studied, which are reduced to the type

$$\begin{aligned} B_4 x = &\widehat{A}^4 x - V\langle \widehat{A}x, F^t\rangle_{H^m} - Y\langle \widehat{A}^2 x, F^t\rangle_{H^m} - S\langle \widehat{A}^3 x, F^t\rangle_{H^m} - \\ &- G\langle \widehat{A}^4 x, F^t\rangle_{H^m} = f, \qquad D(B_4) = D(\widehat{A}^4), \end{aligned} \tag{1.1}$$

where $\widehat{A}$ is one well known correct selfadjoint linear operator, the vectors $V \in \mathbb{H}^m$, $Y \in D(\widehat{A})^m$, $S \in D(\widehat{A}^2)^m$, $G \in D(\widehat{A}^3)^m$, $F \in D(\widehat{A}^4)^m$ and S, Y, V satisfy (3.8).

If an operator B_4 is not biquadratic operator i.e. S, Y, V do not satisfy (3.8), then the correctness and selfadjointness of the problems $B_4 x = f$ can be proved by the method developed in [1]. If however B_4 is biquadratic operator, the proof of the correctness and selfadjointness is much simpler.

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The paper is organized as follows. In Section 2 we recall some basic terminology and notation about operators. In Section 3 we recall the theory of correct and selfadjoint boundary problems with biquadratic operators and prove the proposition 3.8 and, finally, consider some applications of this theory.

2 Terminology and notation

By $\langle x, f\rangle_H$ is denoted the inner product of elements x, f of a complex Hilbert space $\mathbb{H}$. For operators $A : \mathbb{H} \to \mathbb{H}$ we write $D(A)$ and $R(A)$ for the domain and the range of A respectively. An operator $\widehat{A}$ is called *correct* if $R(\widehat{A}) = \mathbb{H}$ and the inverse $\widehat{A}^{-1}$ exists and is continuous on $\mathbb{H}$. Let A be an operator with domain dense in $\mathbb{H}$. The *adjoint* operator $A^* : \mathbb{H} \longrightarrow \mathbb{H}$ of A with domain $D(A^*)$ is defined by the equation $\langle Ax, y\rangle_H = \langle x, A^*y\rangle_H$ for every $x \in D(A)$ and every $y \in D(A^*)$. The domain $D(A^*)$ of A^* consists of all $y \in \mathbb{H}$ for which the functional $x \longmapsto \langle Ax, y\rangle_H$ is continuous on $D(A)$. An operator A is called *selfadjoint* if $A = A^*$. An operator D is called *biquadratic operator* if there exists an operator B such that $D = B^4$. Let $F_i \in \mathbb{H}, i = 1, \ldots, m$. Then $F = (F_1, \ldots, F_m)$ and $AF = (AF_1, \ldots, AF_m)$ are vectors of $\mathbb{H}^m$. Let $\mathcal{F} = (\widehat{A}^{-3}F, \widehat{A}^{-2}F, \widehat{A}^{-1}F, F) = (\widehat{A}^{-3}F_1, \ldots, \widehat{A}^{-3}F_m, \widehat{A}^{-2}F_1, \ldots, \widehat{A}^{-2}F_m, \widehat{A}^{-1}F_1, \ldots,$ $\widehat{A}^{-1}F_m, F_1, \ldots, F_m)$ is a vector of $\mathbb{H}^{4m}$ and $\widehat{A}^{-4} = (\widehat{A}^{-1})^4$. We write F^t and $\langle Ax, F^t\rangle_{H^m}$ for the column vectors $col(F_1, \ldots, F_m)$ and $col(\langle Ax, F_1\rangle_H, \ldots,$ $\langle Ax, F_m\rangle_H)$ respectively. We denote by M^t the transpose matrix of M and by $\langle AF^t, F\rangle_{H^m}$ the $m \times m$ matrix whose i, j-th entry is the inner product $\langle AF_i, F_j\rangle_H$. We also denote by I_m and $[0]_m$ the identity $m \times m$ and the zero $m \times m$ matrix respectively.

3 Correct and selfadjoint problems with biquadratic operators.

We shall make use of the following [2, Lemma 3.3, Theorem 3.4]

Lemma 3.1. *Let the operators $B, B_3 : \mathbb{H} \to \mathbb{H}$ be defined by*

$$Bx = \widehat{A}x - G\langle \widehat{A}x, F^t\rangle_{H^m} = f, \qquad D(B) = D(\widehat{A}), \tag{3.1}$$

$$B_3x = \widehat{A}^3x - Y\langle \widehat{A}x, F^t\rangle_{H^m} - S\langle \widehat{A}^2x, F^t\rangle_{H^m} - G\langle \widehat{A}^3x, F^t\rangle_{H^m} = f,\ D(B_3) = D(\widehat{A}^3), \tag{3.2}$$

where $\widehat{A}$ *is a linear operator on* $\mathbb{H}$, G *is a vector of* $D(\widehat{A}^2)^m$, *the vectors* S, G *satisfy the equations*

$$S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m}, \qquad Y = \widehat{A}S - G\overline{\langle F^t, \widehat{A}S\rangle}_{H^m} \tag{3.3}$$

and the components of the vector $F = (F_1, \ldots, F_m)$ *belong to* $D(\widehat{A}^3)$. *Then* $B_3 = B^3$, *i.e.* B_3 *is a cubic operator.*

Theorem 3.2. *Let the operators* $\widehat{A}, B_3 : \mathbb{H} \to \mathbb{H}$ *and vectors* G, S, Y *be defined as in lemma 3.1. We also assume that* $\widehat{A}$ *is a correct operator,* $G = (\widehat{A}F)C$, *where* C *is a* $m \times m$ *matrix with rank* $C = n \leq m$ *and the components of vector* $\mathcal{F} = (\widehat{A}^{-2}F, \widehat{A}^{-1}F, F)$ $\big($*resp.* $\widehat{A}^2\mathcal{F} = (F, \widehat{A}F, \widehat{A}^2F)\big)$ *are linearly independent elements of* $D(\widehat{A}^3)$ $\big($*resp.* $D(\widehat{A})\big)$. *Then:*
(i) B_3 *is selfadjoint if and only if* C *is Hermitian,*
(ii) $\dim R(B_3 - \widehat{A}^3) = 3n \, (n \leq m)$,
(iii) B_3 *is a correct operator if and only if holds*

$$\det L = \det \big[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C\big] \neq 0. \tag{3.4}$$

(iv) The unique solution, for every $f \in \mathbb{H}$, *of the problem (3.2) is given by*

$$\begin{aligned} x = & B_3^{-1}f = \widehat{A}^{-3}f + \big[\widehat{A}^{-2}F + (\widehat{A}^{-1}F)CL^{-1}\overline{\langle F^t, F\rangle}_{H^m} + \\ & + FW\big]CL^{-1}\langle f, F^t\rangle_{H^m} + \big[\widehat{A}^{-1}F + FCL^{-1}\overline{\langle F^t, F\rangle}_{H^m}\big] \cdot \\ & \cdot CL^{-1}\langle f, \widehat{A}^{-1}F^t\rangle_{H^m} + FCL^{-1}\langle f, \widehat{A}^{-2}F^t\rangle_{H^m}, \end{aligned} \tag{3.5}$$

where $W = CL^{-1}[\overline{\langle \widehat{A}^{-1}F^t, F\rangle}_{H^m} + \overline{\langle F^t, F\rangle}_{H^m} CL^{-1}\overline{\langle F^t, F\rangle}_{H^m}]$.

Next lemma and theorem are Lemma 3.3 and Theorem 3.4 of [3].

Lemma 3.3. *Let the operators* $B, B_4 : \mathbb{H} \to \mathbb{H}$ *be defined by*

$$Bx = \widehat{A}x - G\langle \widehat{A}x, F^t\rangle_{H^m} = f, \qquad D(B) = D(\widehat{A}), \tag{3.6}$$

$$\begin{aligned} B_4x = & \widehat{A}^4x - V\langle \widehat{A}x, F^t\rangle_{H^m} - Y\langle \widehat{A}^2x, F^t\rangle_{H^m} - S\langle \widehat{A}^3x, F^t\rangle_{H^m} - \\ & - G\langle \widehat{A}^4x, F^t\rangle_{H^m} = f, \qquad D(B_4) = D(\widehat{A}^4), \end{aligned} \tag{3.7}$$

where $\widehat{A}$ *is a linear operator on* $\mathbb{H}$, G *is a vector of* $D(\widehat{A}^3)^m$, *the vectors* V, Y, S, G *satisfy the equations*

$$V = \widehat{A}Y - G\overline{\langle F^t, \widehat{A}Y\rangle}_{H^m}, \quad Y = \widehat{A}S - G\overline{\langle F^t, \widehat{A}S\rangle}_{H^m}, \quad S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} \tag{3.8}$$

and the components of the vector $F = (F_1, \ldots, F_m)$ *belong to* $D(\widehat{A}^4)$. *Then* $B_4 = B^4$, *i.e.* B_4 *is an biquadratic operator.*

Theorem 3.4. *Let the operators* $\widehat{A}, B_4 : \mathbb{H} \to \mathbb{H}$ *and vectors* G, S, Y, V *be defined as in lemma 3.3. We also assume that* $\widehat{A}$ *is a correct operator,* $G = (\widehat{A}F)C$, *where* C *is a* $m \times m$ *matrix with rank* $C = n \leq m$ *and the components of vector* $\mathcal{F} = (\widehat{A}^{-3}F, \widehat{A}^{-2}F, \widehat{A}^{-1}F, F)$ $\big($*resp.* $\widehat{A}^3\mathcal{F} = (F, \widehat{A}F, \widehat{A}^2F, \widehat{A}^3F)\big)$ *are linearly independent elements of* $D(\widehat{A}^4)$ $\big($*resp.* $D(\widehat{A})\big)$. *Then:*
(i) B_4 *is a selfadjoint operator if and only if* C *is Hermitian,*
(ii) B_4 *is a correct operator if and only if holds*

$$\det L = \det\left[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C\right] \neq 0. \tag{3.9}$$

(iii) If B_4 *is a correct operator, then* $\dim R(B_4 - \widehat{A}^4) = 4n\,(n \leq m)$,
(iv) The unique solution of the problem (3.7), where B_4 *is correct, for every* $f \in \mathbb{H}$ *is given by*

$$\begin{aligned} x =& B_4^{-1}f = \widehat{A}^{-4}f + \Big[\widehat{A}^{-3}F + (\widehat{A}^{-2}F)CL^{-1}\overline{\langle F^t, F\rangle}_{H^m} + (\widehat{A}^{-1}F)W + FCL^{-1}\cdot \\ &\cdot\big(\overline{\langle \widehat{A}^{-2}F^t, F\rangle}_{H^m} + \overline{\langle \widehat{A}^{-1}F^t, F\rangle}_{H^m} CL^{-1}\overline{\langle F^t, F\rangle}_{H^m} + \overline{\langle F^t, F\rangle}_{H^m} W\big)\Big] CL^{-1}\cdot \\ &\cdot\langle f, F^t\rangle_{H^m} + \big[\widehat{A}^{-2}F + (\widehat{A}^{-1}F)CL^{-1}\overline{\langle F^t, F\rangle}_{H^m} + FW\big]CL^{-1}\langle f, \widehat{A}^{-1}F^t\rangle_{H^m} \\ &+ [\widehat{A}^{-1}F + FCL^{-1}\overline{\langle F^t, F\rangle}_{H^m}]CL^{-1}\langle f, \widehat{A}^{-2}F^t\rangle_{H^m} + FCL^{-1}\langle f, \widehat{A}^{-3}F^t\rangle_{H^m}, \end{aligned} \tag{3.10}$$

where $W = CL^{-1}[\overline{\langle \widehat{A}^{-1}F^t, F\rangle}_{H^m} + \overline{\langle F^t, F\rangle}_{H^m} CL^{-1}\overline{\langle F^t, F\rangle}_{H^m}]$.

Bellow $H^i(0,1)$ denote the Sobolev spaces of all complex functions of $L_2(0,1)$ which have generalized derivatives up to i -th order respectively Lebesque integrable, $i = 1,2,3,4$ and we have used the programs Derive and Mathematica 6 for computing of integrals and some complex expressions.

It is easy to verify that the operator $\widehat{A} : L_2(0,1) \to L_2(0,1)$ defined by

$$\widehat{A}u = u'' = f, \quad D(\widehat{A}) = \{u(t) \in H^2(0,1) : u(1) = u'(0) = 0\} \tag{3.11}$$

is correct and selfadjoint and the unique solution u of the problem (3.11) is given by the formula

$$u(t) = \widehat{A}^{-1}f(t) = \int_0^t (t-x)f(x)dx + \int_0^1 (x-1)f(x)dx \quad \text{for all } f \in H. \tag{3.12}$$

Proposition 3.5. *Let the operator* $\widehat{A}$ *defined by (3.11). Then the operator* $\widehat{A}^2 : L_2(0,1) \to L_2(0,1)$ *defined by*

$$\widehat{A}^2u = u^{(4)} = f, \quad D(\widehat{A}^2) = \{u \in H^4(0,1) : u(1) = u'(0) = u''(1) = u'''(0) = 0\} \tag{3.13}$$

is correct and selfadjoint and for every $f \in L_2(0,1)$ the unique solution u of the problem (3.13) is given by the formula

$$u(t) = \widehat{A}^{-2} f(t) = \frac{1}{6}\int_0^t (t-x)^3 f(x)dx + \qquad (3.14)$$
$$+\frac{1}{6}\int_0^1 (x-1)^3 f(x)dx + \frac{t^2-1}{2}\int_0^1 (x-1)f(x)dx$$

Proof. Correctness and selfadjointness of $\widehat{A}$ implies correctness and selfadjointness of $\widehat{A}^2$. Now we will prove the formula (3.14). Let $y(x) = \widehat{A}^{-1} f(x)$. Then by (3.21) and Fubini's theorem we have

$$\widehat{A}^{-2} f(t) = \widehat{A}^{-1}\big(\widehat{A}^{-1} f(t)\big) = \widehat{A}^{-1} y(t) = \int_0^t (t-x)y(x)dx + \int_0^1 (x-1)y(x)dx =$$
$$= \int_0^t (t-x)\Big[\int_0^x (x-z)f(z)dz + \int_0^1 (z-1)f(z)dz\Big]dx +$$
$$+ \int_0^1 (x-1)\Big[\int_0^x (x-z)f(z)dz + \int_0^1 (z-1)f(z)dz\Big]dx =$$
$$= \int_0^t dx \int_0^x (x-z)(t-x)f(z)dz + \int_0^t (t-x)dx \int_0^1 (z-1)f(z)dz +$$
$$+ \int_0^1 dx \int_0^x (x-z)(x-1)f(z)dz + \int_0^1 (x-1)dx \int_0^1 (z-1)f(z)dz =$$
$$= \int_0^t dz \int_z^t (x-z)(t-x)f(z)dx + \frac{t^2-2t}{2}\int_0^1 (z-1)f(z)dz +$$
$$+ \int_0^1 dz \int_z^1 (x-z)(x-1)f(z)dx + \frac{2t-1}{2}\int_0^1 (z-1)f(z)dz =$$
$$= \int_0^t f(z)dz \int_z^t (x-z)(t-x)dx + \frac{t^2-1}{2}\int_0^1 (z-1)f(z)dz +$$
$$+ \int_0^1 f(z)dz \int_z^1 (x-z)(x-1)dx = \frac{1}{6}\int_0^t (t-z)^3 f(z)dz +$$
$$+ \frac{t^2-1}{2}\int_0^1 (z-1)f(z)dz + \frac{1}{6}\int_0^1 (z-1)^3 f(z)dz =$$
$$= \frac{1}{6}\int_0^t (t-z)^3 f(z)dz + \frac{1}{6}\int_0^1 (z-1)^3 f(z)dz + \frac{t^2-1}{2}\int_0^1 (z-1)f(z)dz$$

which gives (3.14). □

Proposition 3.6. *Let the operator $\widehat{A}$ defined by (3.11). Then the operator $\widehat{A}^3 : L_2(0,1) \to L_2(0,1)$ defined by*

$$\widehat{A}^3 u = u^{(6)} = f, \qquad (3.15)$$
$$D(\widehat{A}^3) = \{u \in H^6(0,1) : u(1) = u'(0) = u''(1) = u'''(0) = u^{(4)}(1) = u^{(5)}(0) = 0\}$$

is correct and selfadjoint and for every $f \in L_2(0,1)$ the unique solution u of the problem (3.15) is given by the formula

$$u(t) = \widehat{A}^{-3} f(t) = \frac{1}{120}\int_0^t (t-x)^5 f(x)dx + \frac{1}{120}\int_0^1 (x-1)^5 f(x)dx + \\ + \frac{t^2-1}{12}\int_0^1 (x-1)^3 f(x)dx + \frac{t^4-6t^2+5}{24}\int_0^1 (x-1)f(x)dx \tag{3.16}$$

Proof. Correctness and selfadjointness of $\widehat{A}$ implies correctness and selfadjointness of $\widehat{A}^3$. Now we will prove the formula (3.16). Let $y(x) = \widehat{A}^{-2} f(x)$. Then by (3.21), (3.14) and Fubini's theorem we have

$$\widehat{A}^{-3} f(t) = \widehat{A}^{-1}\big(\widehat{A}^{-2} f(t)\big) = \widehat{A}^{-1} y(t) = \int_0^t (t-x)y(x)dx + \int_0^1 (x-1)y(x)dx =$$
$$= \int_0^t (t-x)\Big[\frac{1}{6}\int_0^x (x-z)^3 f(z)dz + \frac{1}{6}\int_0^1 (z-1)^3 f(z)dz + \frac{x^2-1}{2}\int_0^1 (z-1)f(z)dz\Big]dx$$
$$+ \int_0^1 (x-1)\Big[\frac{1}{6}\int_0^x (x-z)^3 f(z)dz + \frac{1}{6}\int_0^1 (z-1)^3 f(z)dz + \frac{x^2-1}{2}\int_0^1 (z-1)f(z)dz\Big]dx$$
$$= \frac{1}{6}\int_0^t (t-x)dx\int_0^x (x-z)^3 f(z)dz + \frac{1}{6}\int_0^t (t-x)dx\int_0^1 (z-1)^3 f(z)dz +$$
$$+ \frac{1}{2}\int_0^t (t-x)(x^2-1)dx\int_0^1 (z-1)f(z)dz + \frac{1}{6}\int_0^1 (x-1)dx\int_0^x (x-z)^3 f(z)dz +$$
$$+ \frac{1}{6}\int_0^1 (x-1)dx\int_0^1 (z-1)^3 f(z)dz + \frac{1}{2}\int_0^1 (x-1)(x^2-1)dx\int_0^1 (z-1)f(z)dz =$$
$$= \frac{1}{6}\int_0^t f(z)dz\int_z^t (t-x)(x-z)^3 dx + \frac{t^2}{12}\int_0^1 (z-1)^3 f(z)dz +$$
$$+ \frac{t^4-6t^2}{24}\int_0^1 (z-1)f(z)dz + \frac{1}{6}\int_0^1 f(z)dz\int_z^1 (x-1)(x-z)^3 dx - \frac{1}{12}\int_0^1 (z-1)^3 f(z)dz$$
$$+ \frac{5}{24}\int_0^1 (z-1)f(z)dz = \frac{1}{120}\int_0^t (t-z)^5 f(z)dz + \frac{1}{120}\int_0^1 (z-1)^5 f(z)dz +$$
$$+ \frac{t^2-1}{12}\int_0^1 (z-1)^3 f(z)dz + \frac{t^4-6t^2+5}{24}\int_0^1 (z-1)f(z)dz$$

which gives (3.16). □

Example 3.7. The operator $B_1 : L_2(0,1) \to L_2(0,1)$ which corresponds to the problem

$$\begin{aligned} B_1 u =& u^{(6)} - \frac{21}{7688}(649t^4 - 4638t^2 + 4175)\int_0^1 (x^6 - 15x^4 + 75x^2 - 61)u''(x)dx- \\ &-\frac{63}{124}(2t^4 - 15t^2 + 13)\int_0^1 (x^5 - 10x^3 + 25x)u'''(x)dx - \frac{21}{992}(t^4 - 6t^2 + 5)\cdot \\ &\cdot\int_0^1 (x^6 - 15x^4 + 75x^2 - 61)u^{(6)}(x)dx = f(t), \quad D(B_1) = D(\widehat{A}^3) \quad (3.17) \end{aligned}$$

is correct and selfadjoint and the unique solution of (3.17), for each $f \in L_2(0,1)$, is given by the formula

$$\begin{aligned} u(t) =& \widehat{A}^{-3}f(t) + \Big[\frac{1}{5040}(t^{10} - 45t^8 + 1050t^6 - 12810t^4 + 62325t^2 - 50521)+ \\ &+\frac{1}{56}(t^8 - 28t^6 + 350t^4 - 1708t^2 + 1385)\frac{7}{9920}\frac{77}{353869}\frac{5592064}{3003} + (t^6 - 15t^4 + \\ &+75t^2 - 61)\frac{7}{9920}\frac{77}{353869}\Big[\Big(\frac{-237969664}{315315} + \frac{5592064}{3003}\frac{7}{9920}\frac{77}{353869}\frac{5592064}{3003}\Big]\Big]\cdot \\ &\cdot\frac{7}{9920}\frac{77}{353869}\int_0^1 (x^6 - 15x^4 + 75x^2 - 61)f(x)dx+, \quad (3.18) \end{aligned}$$

where $\widehat{A}^{-3}f(t)$ is defined by (3.16).

Proof. We refer to theorem 3.2. If we compare equation (3.17) with equation (3.2) it is natural to take $\widehat{A}^3 u = u^{(6)}$ with $D(\widehat{A}^3) = D(B_1)$, $m = 1$, $F = t^6 - 15t^4 + 75t^2 - 61$. Then we can take $\widehat{A}$ to be defined by (3.11), $\widehat{A}^2$ by (3.13). It is evident that $F \in D(\widehat{A}^3)$, $\widehat{A}F = 30(t^4 - 6t^2 + 5)$, $\widehat{A}^2 F = 12(t^2 - 1)$, and that $\langle \widehat{A}u, F\rangle_H = \int_0^1 u''(x)(x^6 - 15x^4 + 75x^2 - 61)dx$, $\langle \widehat{A}^2 u, F\rangle_H = \int_0^1 u^{(4)}(x)(x^6 - 15x^4 + 75x^2 - 61)dx$, $\langle \widehat{A}^3 u, F\rangle_H = \int_0^1 u^{(6)}(x)(x^6 - 15x^4 + 75x^2 - 61)dx$. By integrating by parts we have $\langle \widehat{A}^2 u, F\rangle_H = \int_0^1 u^{(4)}(x)(x^6 - 15x^4 + 75x^2 - 61)dx = -6\int_0^1 (x^5 - 10x^3 + 25x)u'''(x)$. Then $\int_0^1 (x^5 - 10x^3 + 25x)u'''(x) = -\frac{1}{6}\langle \widehat{A}^2 u, F\rangle_H$. Replacing these elements in (3.17) we get:

$$\begin{aligned} B_1 u =& \widehat{A}^3 u - \frac{21}{7688}(649t^4 - 4638t^2 + 4175)\langle \widehat{A}u, F\rangle_H + \quad (3.19) \\ &+\frac{21}{248}(2t^4 - 15t^2 + 13)\langle \widehat{A}^2 u, F\rangle_H - \frac{21}{992}(t^4 - 6t^2 + 5)\langle \widehat{A}^3 u, F\rangle_H = f(t). \end{aligned}$$

Again, comparing (3.31) with (3.2)) we take $Y = \frac{21}{7688}(649t^4 - 4638t^2 + 4175)$, $S = -\frac{21}{248}(2t^4 - 15t^2 + 13)$ and $G = \frac{21}{992}(t^4 - 6t^2 + 5)$. It is evident that $G \in D(\widehat{A}^2)$ and $F, \widehat{A}F, \widehat{A}^2 F$ are linearly independent elements of $D(\widehat{A})$. By using the program Derive and Mathematica 6 we find $\widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} = -\frac{21}{248}(2t^4 - 15t^2 + 13) = S$ and $\widehat{A}S - G\overline{\langle F^t, \widehat{A}S\rangle}_{H^m} = \frac{21}{7688}(649t^4 - 4638t^2 + 4175) = Y$. The last two equalities, by lemma 3.1, show that the operator B_1 is cubic, i.e. $B_1 = B_3$. From $G = (\widehat{A}F)C$ it follows $\frac{21}{992}(t^4 - 6t^2 + 5) = 30(t^4 - 6t^2 + 5)C$. This equation implies that $C = 7/9920$. Again, using Derive we take $\langle F^t, F\rangle_H = 5592064/3003$, $\langle \widehat{A}F^t, F\rangle_H = -353792/77$. By theorem 3.2 the operator B_1 is correct and selfadjoint since $C = 7/9920$ is a real number and $\det L = \det[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C] = 1 + 353792/77 = 353869/77 \neq 0$. Hence $L^{-1} = 77/353869$. If we substitute in (3.21) and (3.14) $f = F = t^6 - 15t^4 + 75t^2 - 61$, we receive $\widehat{A}^{-1}F = \frac{1}{56}(t^8 - 28t^6 + 350t^4 - 1708t^2 + 1385)$ and $\widehat{A}^{-2}F = \frac{1}{5040}(t^{10} - 45t^8 + 1050t^6 - 12810t^4 + 62325t^2 - 50521)$. Then $\langle f, \widehat{A}^{-1}F\rangle_H = \frac{1}{56}\int_0^1 (x^8 - 28x^6 + 350x^4 - 1708x^2 + 1385)f(x)dx$, $\langle f, \widehat{A}^{-2}F\rangle_H = \frac{1}{5040}\int_0^1 (x^{10} - 45x^8 + 1050x^6 - 12810x^4 + 62325x^2 - 50521)f(x)dx$ and using the program Derive we have $\langle \widehat{A}^{-1}F, F\rangle_H = -237969664/315315$. As a result of this and (3.5), (3.16) we get the solution (3.18) of the problem (3.17). □

We recall [1, p.780] that the operator $\widehat{A} : L_2(0,1) \to L_2(0,1)$ defined by

$$\widehat{A}u = iu' = f, \quad D(\widehat{A}) = \{u(t) \in H^1(0,1) : u(0) + u(1) = 0\} \tag{3.20}$$

is correct and selfadjoint and the unique solution u of the problem (3.20) is given by the formula

$$u(t) = \widehat{A}^{-1}f(t) = \frac{i}{2}\int_0^1 f(x)dx - i\int_0^t f(x)dx \quad \text{for all } f \in H. \tag{3.21}$$

Then [2, p.] the operator $\widehat{A}^2$ defined by

$$\begin{aligned} &\widehat{A}^2 u = -u'' = f, \\ &D(\widehat{A}^2) = \{u \in H^2(0,1) : u(0) + u(1) = 0,\ u'(0) + u'(1) = 0\}, \end{aligned} \tag{3.22}$$

is correct and selfadjoint and for every $f \in L_2(0,1)$ the unique solution u of the problem (3.22) is given by the formula

$$u(t) = \widehat{A}^{-2}f(t) = -\int_0^t (t-x)f(x)dx + \frac{1}{4}\int_0^1 (2t - 2x + 1)f(x)dx. \tag{3.23}$$

Also [2, Proposition 3.6] the operator $\widehat{A}^3$ defined by

$$\begin{aligned} &\widehat{A}^3 u = -iu''' = f, \\ &D(\widehat{A}^3) = \{u \in H^3(0,1) : u(0) + u(1) = 0,\ u'(0) + u'(1) = 0,\ u''(0) + u''(1) = 0\}, \end{aligned} \tag{3.24}$$

is correct and selfadjoint and for every $f \in L_2(0,1)$ the unique solution u of the problem (3.24) is given by the formula

$$u(t) = \widehat{A}^{-3}f(t) = \frac{i}{2}\int_0^t (t-x)^2 f(x)dx - \frac{i}{4}\int_0^1 (t-x)(t-x+1)f(x)dx. \tag{3.25}$$

Proposition 3.8. *Let the operator $\widehat{A}$ defined by (3.20). Then the operator $\widehat{A}^4 : L_2(0,1) \to L_2(0,1)$ defined by*

$$\widehat{A}^4 u = u^{(4)} = f, \qquad D(\widehat{A}^4) = \{u \in H^4(0,1) : u^{(k)}(0) + u^{(k)}(1) = 0,\ k = 0,1,2,3\}, \tag{3.26}$$

$\widehat{A}^4$ is correct and selfadjoint and for every $f \in L_2(0,1)$ the unique solution u of the problem (3.26) is given by the formula

$$u(t) = \widehat{A}^{-4} f(t) = \frac{1}{6}\int_0^t (t-x)^3 f(x)dx + \frac{1}{48}\int_0^1 \left[4x^3 - 6x^2(2t+1) + 12tx(t+1) - 4t^3 - 6t^2 + 1\right] f(x)dx \tag{3.27}$$

Proof. Correctness and selfadjointness of $\widehat{A}$ implies correctness and selfadjointness of $\widehat{A}^3$. Now we will prove the formula (3.27). Let $y(x) = \widehat{A}^{-3} f(x)$. Then by (3.21), (3.25) and Fubini's theorem we have

$$\widehat{A}^{-4} f(t) = \widehat{A}^{-1}\big(\widehat{A}^{-3} f(t)\big) = \widehat{A}^{-1} y(t) = \frac{i}{2}\int_0^1 y(z)dz - i\int_0^t y(z)dz =$$
$$= \frac{i}{2}\int_0^1 \left[\frac{i}{2}\int_0^z (z-x)^2 f(x)dx - \frac{i}{4}\int_0^1 (z-x)(z-x+1) f(x)dx\right]dz -$$
$$- i\int_0^t \left[\frac{i}{2}\int_0^z (z-x)^2 f(x)dx - \frac{i}{4}\int_0^1 (z-x)(z-x+1) f(x)dx\right]dz =$$
$$= \frac{-1}{4}\int_0^1 dz \int_0^z (z-x)^2 f(x)dx + \frac{1}{8}\int_0^1 dz \int_0^1 (z-x)(z-x+1) f(x)dx +$$
$$+ \frac{1}{2}\int_0^t dz \int_0^z (z-x)^2 f(x)dx - \frac{1}{4}\int_0^t dz \int_0^1 (z-x)(z-x+1) f(x)dx =$$
$$= -\frac{1}{4}\int_0^1 f(x)dx \int_x^1 (z-x)^2 dz + \frac{1}{8}\int_0^1 f(x)dx \int_0^1 (z-x)(z-x+1)dz +$$
$$+ \frac{1}{2}\int_0^t f(x)dx \int_x^t (z-x)^2 dz - \frac{1}{4}\int_0^1 f(x)dx \int_0^t (z-x)(z-x+1)dz +$$
$$= \frac{1}{12}\int_0^1 (x-1)^3 f(x)dx + \frac{1}{48}\int_0^1 (6x^2 - 12x + 5) f(x)dx + \frac{1}{6}\int_0^t (t-x)^3 f(x)dx -$$
$$- \frac{t}{24}\int_0^1 \left[6x^2 - 6x(t+1) + 2t^2 + 3t\right] f(x)dx = \frac{1}{6}\int_0^t (t-x)^3 f(x)dx +$$
$$+ \frac{1}{48}\int_0^1 \left[4x^3 - 6x^2(2t+1) + 12tx(t+1) - 4t^3 - 6t^2 + 1\right] f(x)dx \tag{3.28}$$

which gives (3.27). □

Example 3.9. The operator $B_1 : L_2(0,1) \to L_2(0,1)$ which corresponds to the problem

$$B_1 u = u^{(4)} + 80\Big\{\frac{620}{21}(t^2-t) - \frac{65654570}{63^3}(t^4-2t^3+t) - i\Big[6t-3-\frac{310^2}{63^2}(4t^3- \\ -6t^2+1)\Big]\Big\}\int_0^1 (2x^5-5x^4+5x^2-1)u\,'(x)dx + 80\Big\{\frac{155}{63}(4t^3-6t^2+1)- \\ -i\Big[3t^2-3t-\frac{310^2}{63^2}(t^4-2t^3+t)\Big]\Big\}\int_0^1 u\,''(x)(2x^5-5x^4+5x^2-1)dx- \\ -20i\Big[\frac{620i}{63}(t^4-2t^3+t)+4t^3-6t^2+1\Big]\int_0^1 u\,'''(x)(2x^5-5x^4+5x^2-1)dx+ \\ +200i(t^4-2t^3+t)\int_0^1 u\,'''(x)(x^4-2x^3+x)dx = f(t), \quad D(B_1) = D(\widehat{A}^4) \tag{3.29}$$

is correct and selfadjoint and the unique solution of (3.29), for each $f \in L_2(0,1)$, is given by the formula

$$u(t) = \widehat{A}^{-4}f(t) + \frac{2}{3}\Big[\frac{i}{56}(t^8-4t^7+14t^5-28t^3+17t) - \frac{691}{38808}(8t^7-28t^6+ \\ +70t^4-84t^2+17) - \frac{691^2 i}{693^2}(t^6-3t^5+5t^3-3t) + \frac{5461}{18018}(2t^5-5t^4+ \\ +5t^2-1)\Big]\int_0^1 (2x^5-5x^4+5x^2-1)f(x)dx + \frac{2i}{9}\Big[\frac{-1}{56}(8t^7-28t^6+70t^4- \\ -84t^2+17) - \frac{691i}{693}(t^6-3t^5+5t^3-3t) + \frac{3\cdot 691^2}{693^2}(2t^5-5t^4+5t^2-1)\Big]\cdot \\ \cdot\int_0^1 (x^6-3x^5+5x^3-3x)f(x)dx + \frac{2}{504}\Big[-i(t^6-3t^5+5t^3-3t)+ \\ +\frac{691}{231}(2t^5-5t^4+5t^2-1)\Big]\int_0^1 (8x^7-28x^6+70x^4-84x^2+17)f(x)dx- \\ -\frac{2i}{168}(2t^5-5t^4+5t^2-1)\int_0^1 (x^8-4x^7+14x^5-28x^3+17x)f(x)dx, \tag{3.30}$$

where $\widehat{A}^{-4}f(t)$ is defined by (3.27).

Proof. We refer to theorem 3.4. If we compare equation (3.29) with equation (3.7) it is natural to take $\widehat{A}^4 u = u^{(4)}$ with $D(\widehat{A}^4) = D(B_1)$, $m = 1$, $F = 2t^5 - 5t^4 + 5t^2 - 1$. Then we can take $\widehat{A}$ to be defined by (3.20), $\widehat{A}^2$ by (3.22), $\widehat{A}^3$ by (3.24). It is evident that $F \in D(\widehat{A}^4)$, $\widehat{A}F = 10i(t^4 - 2t^3 + t)$, $\widehat{A}^2 F = -10(4t^3-6t^2+1)$, $\widehat{A}^3 F = -120i(t^2-t)$, $\widehat{A}^4 F = 120(2t-1)$, and that $\langle \widehat{A}u, F\rangle_H = \int_0^1 iu'(x)(2t^5-5t^4+5t^2-1)dx$, $\langle \widehat{A}^2 u, F\rangle_H = -\int_0^1 u''(x)(2t^5-5t^4+$

$5t^2-1)dx$, $\langle \widehat{A}^3u, F\rangle_H = -i\int_0^1 u'''(x)(2t^5-5t^4+5t^2-1)dx$, $\langle \widehat{A}^4u, F\rangle_H = \int_0^1 u^{(4)}(x)(2x^5-5x^4+5x^2-1)dx$. Then $\int_0^1 u'(x)(2x^5-5x^4+5x^2-1)dx = -i\langle \widehat{A}u, F\rangle_H$, $\int_0^1 u''(x)(2x^5-5x^4+5x^2-1)dx = -\langle \widehat{A}^2u, F\rangle_H$, $\int_0^1 u'''(x)(2x^5-5x^4+5x^2-1)dx = i\langle \widehat{A}^3u, F\rangle_H$. By integrating by parts we have $\langle \widehat{A}^4u, F\rangle_H = -10\int_0^1 u'''(x)(x^4-2x^3+x)dx$. Then $\int_0^1 u'''(x)(x^4-2x^3+x)dx = -\frac{1}{10}\langle \widehat{A}^4u, F\rangle_H$. Replacing these elements in (3.29) we get:

$$\begin{aligned} B_1u =& \widehat{A}^4u - 80i\Big\{\frac{620}{21}(t^2-t) - \frac{65654570}{63^3}(t^4-2t^3+t) - i\Big[6t-3-\frac{310^2}{63^2}(4t^3- \\ & -6t^2+1)\Big]\Big\}\langle \widehat{A}u, F\rangle_H - 80\Big\{\frac{155}{63}(4t^3-6t^2+1) - i\Big[3t^2-3t-\frac{310^2}{63^2}(t^4-2t^3+ \\ & +t)\Big]\Big\}\langle \widehat{A}^2u, F\rangle_H + 20\Big[\frac{620i}{63}(t^4-2t^3+t)+4t^3-6t^2+1\Big]\langle \widehat{A}^3u, F\rangle_H - \\ & -20i(t^4-2t^3+t)\langle \widehat{A}^4u, F\rangle_H = f(t). \end{aligned} \tag{3.31}$$

Again, comparing (3.31) with (3.7)) we get
$V = 80i\Big\{\frac{620}{21}(t^2-t) - \frac{65654570}{63^3}(t^4-2t^3+t) - i\Big[6t-3-\frac{310^2}{63^2}(4t^3-6t^2+1)\Big]\Big\}$,
$Y = 80\Big\{\frac{155}{63}(4t^3-6t^2+1) - i\Big[3t^2-3t-\frac{310^2}{63^2}(t^4-2t^3+t)\Big]\Big\}$,
$S = -20\Big[\frac{620i}{63}(t^4-2t^3+t)+4t^3-6t^2+1\Big]$ and $G = 20i(t^4-2t^3+t)$.
It is evident that $G \in D(\widehat{A}^3)$ and $F, \widehat{A}F, \widehat{A}^2F, \widehat{A}^3F$ are linearly independent elements of $\overline{D(\widehat{A})}$. By using the program Derive and Mathematica 6 we find
$\widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} = -20(4t^3-6t^2+1) - 20i(t^4-2t^3+t)\frac{620}{63} = S$,
$\widehat{A}S - G\overline{\langle F^t, \widehat{A}S\rangle}_{H^m} = -20i\Big[\frac{620i}{63}(4t^3-6t^2+1)+12(t^2-t)\Big] - 20i(t^4-2t^3+ +t)\big(-\frac{620^2}{63^2}\big) = Y$ and $\widehat{A}Y - G\overline{\langle F^t, \widehat{A}Y\rangle}_{H^m} = 80i\Big\{\frac{155}{63}(12t^2-12t) - i\Big[6t-3- -\frac{310^2}{63^2}(4t^3-6t^2+1)\Big]\Big\} - 20i(t^4-2t^3+t)\frac{262618280}{63^3} = V$. The last three equalities, by lemma 3.3, show that the operator B_1 is biquadratic, i.e. $B_1 = B_4$. From $G = (\widehat{A}F)C$ it follows $20i(t^4-2t^3+t) = 10i(t^4-2t^3+t)C$. This equation implies that $C = 2$. Again, using Derive and Mathematica 6 we get $\langle F^t, F\rangle_H = \frac{691}{1386}$, $\langle \widehat{A}F^t, F\rangle_H = 0$. By theorem 3.4 the operator B_1 is correct and selfadjoint since $C = 2$ is a real number and $\det L = \det[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m}C] = 1-0 = 1 \neq 0$. Hence $L^{-1} = 1$. If we substitute in (3.21), (3.23) and (3.25) $f = F = 2x^5-5x^4+5x^2-1$, we receive $\widehat{A}^{-1}F = -\frac{i}{3}(t^6-3t^5+5t^3-3t)$, $\widehat{A}^{-2}F = -\frac{1}{168}(8t^7-28t^6+70t^4-84t^2+17)$ and $\widehat{A}^{-3}F = \frac{i}{168}(t^8-4t^7+14t^5-28t^3+17t)$ Then $\langle f, \widehat{A}^{-1}F\rangle_H = -\frac{i}{3}\int_0^1(x^6-3x^5+5x^3-3x)f(x)dx$, $\langle f, \widehat{A}^{-2}F\rangle_H = -\frac{1}{168}\int_0^1(8x^7-28x^6+70x^4-84x^2+17)f(x)dx$ and $\langle f, \widehat{A}^{-3}F\rangle_H = \frac{i}{168}\int_0^1(x^8-4x^7+14x^5-28x^3+17x)f(x)dx$ using the program Derive and Mathematica 6 we have $\langle \widehat{A}^{-1}F, F\rangle_H = 0$, $\langle \widehat{A}^{-2}F, F\rangle_H = \frac{5461}{108108}$ and from (3.10) $W = \frac{4\cdot 691^2}{1386^2}$. As a result of this and (3.10) we get the solution

(3.30) of the problem (3.29). □

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