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CORRECT AND SELF-ADJOINT PROBLEMS FOR BIQUADRATIC OPERATORS

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In this paper, we continue a series of previous articles and present a simple method of proving the correctness and self-adjointness of operators of the form B^4 corresponding to some boundary value problems. We also give representations for the unique solutions of these problems. The algorithm is easy to implement via computer algebra systems. In our examples, Derive *and* Mathematica *were used. Bibliography:* 13 *titles.*

1. Introduction

An important tool in creating correct operators and solving boundary value problems containing differential or integro-differential equations is the theory of correct extensions of minimal operators. Correct extensions of densely defined minimal operators in Banach and Hilbert spaces were investigated by M. I. Vishik [2], A. A. Dezin [7], M. Otelbaev [8], R. Oinarov [9], and many others. Self-adjoint extensions of a densely defined minimal symmetric operator A_0 were studied by a number of authors, such as M. G. Krein [1], E. A. Coddington, A. Dijksma [3, 4], V. I. Gorbachuk and M. L. Gorbachuk [5], A. N. Kochubei [6], and many others. Correct self-adjoint and positive extensions of nondensely defined minimal symmetric operators were considered in [10]. Correct self-adjoint problems for quadratic and cubic operators were investigated in [11, 12] and [13].

In this paper, using the operator B defined by

$$Bx = \widehat{A}x - (\widehat{A}F)C\langle \widehat{A}x, F^t\rangle_{H^m} = f, \quad D(B) = D(\widehat{A}),$$

where $\widehat{A}$ is a certain well-known correct self-adjoint operator and C is an $m \times m$ matrix, we investigate the operator B_4 corresponding to the boundary problem:

$$\begin{aligned} &B_4 x = \widehat{A}^4 x - V\langle \widehat{A}x, F^t\rangle_{H^m} - Y\langle \widehat{A}^2 x, F^t\rangle_{H^m} - S\langle \widehat{A}^3 x, F^t\rangle_{H^m} - G\langle \widehat{A}^4 x, F^t\rangle_{H^m} = f,\\ &D(B_4) = D(\widehat{A}^4), \end{aligned} \tag{1.1}$$

where $V \in \mathbb{H}^m$, $Y \in D(\widehat{A})^m$, $S \in D(\widehat{A}^2)^m$, $G \in D(\widehat{A}^3)^m$, $F \in D(\widehat{A}^4)^m$, and V, S, Y satisfy (3.12).

We show that the operator B_4 is biquadratic, i.e., $B_4 = B^4$, and prove a criterion for the correctness and self-adjointness of the problem (1.1) in terms of the matrices C. We also give representations for the unique solution of this problem that are substantially simpler than in the general case of non-biquadratic operators. Note that the self-adjointness of B_4 can be proved by the more general method developed in [2] or [3]. But here we do not need the full strength of this method and use a simpler and straightforward way.

The paper is organized as follows. In Sec. 2, we recall some basic terminology and notation related to operators. In Sec. 3, we prove the main result and give an example of an integro-differential equation that shows the usefulness of our results.

2. Terminology and notation

By $\langle x, f\rangle_H$ we denote the inner product of elements x, f of a complex Hilbert space $\mathbb{H}$. For an operator $A : \mathbb{H} \to \mathbb{H}$, we write $D(A)$ and $R(A)$ for the domain and the range of A, respectively. An operator $\widehat{A}$ is called *correct* if $R(\widehat{A}) = \mathbb{H}$ and the inverse $\widehat{A}^{-1}$ exists and is continuous. An operator B_1 is called *biquadratic* if there exists an operator B such that $B_1 = B^4$. Let A be an operator with domain $D(A)$ dense in $\mathbb{H}$. The *adjoint* operator $A^* : \mathbb{H} \longrightarrow \mathbb{H}$ of A with domain $D(A^*)$ is defined by the equation $\langle Ax, y\rangle_H = \langle x, A^*y\rangle_H$ for every $x \in D(A)$ and every $y \in D(A^*)$. The domain $D(A^*)$ of A^* consists of all $y \in \mathbb{H}$ for which the functional $x \longmapsto \langle Ax, y\rangle_H$ is continuous on $D(A)$. An operator A is called *self-adjoint* if $A = A^*$. If an operator $B : H \to H$ is correct (respectively, self-adjoint), then we say that *the problem* $Bx = f$ *is correct* (respectively, *self-adjoint*). Let $F_i, g_i \in \mathbb{H}, i = 1, \ldots, m$. Then $F = (F_1, \ldots, F_m)$, $G = (g_1, \ldots, g_m)$, and $AF = (AF_1, \ldots, AF_m)$ are vectors of $\mathbb{H}^m$.

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Let $\mathcal{F} = (\widehat{A}^{-3}F, \widehat{A}^{-2}F, \widehat{A}^{-1}F, F) = (\widehat{A}^{-3}F_1, \ldots, \widehat{A}^{-3}F_m, \widehat{A}^{-2}F_1, \ldots, \widehat{A}^{-2}F_m, \widehat{A}^{-1}F_1, \ldots, \widehat{A}^{-1}F_m, F_1, \ldots, F_m)$ (it is a vector of $\mathbb{H}^{4m}$) and $\widehat{A}^{-4} = (\widehat{A}^{-1})^4$. We also write F^t and $\langle Ax, F^t\rangle_{H^m}$ for the column vectors $\mathrm{col}(F_1, \ldots, F_m)$ and $\mathrm{col}(\langle Ax, F_1\rangle_H, \ldots, \langle Ax, F_m\rangle_H)$, respectively. We denote by $\overline{M}$ (respectively, M^t) the complex conjugate (respectively, transpose) matrix of M and by $\langle G^t, F\rangle_{H^m}$ the $m \times m$ matrix whose (i, j)th entry is the inner product $\langle g_i, F_j\rangle_H$. Note that $\langle G^t, F\rangle_{H^m}$ determines the matrix inner product and has the following properties: $\langle CG^t, F\rangle_H = C\langle G^t, F\rangle_H$, $\langle G^t, FC\rangle_H = \langle G^t, F\rangle_H\overline{C}$, $\langle G^t, F\rangle_H = \overline{\langle F^t, G\rangle}_H^t$, where C is a constant $m \times m$ matrix. It is obvious that $\langle f, F^t\rangle_H = \overline{\langle F^t, f\rangle}_H$. We also denote by I_m and $[0]_m$ the identity and zero $m \times m$ matrices, respectively.

3. Correct and self-adjoint problems for biquadratic operators

The following theorem is Theorem 3.1 of [13].

Theorem 3.1. *Let $B : \mathbb{H} \to \mathbb{H}$ and*

$$Bx = \widehat{A}x - (\widehat{A}F)C\langle \widehat{A}x, F^t\rangle_{H^m} = f, \quad D(B) = D(\widehat{A}), \tag{3.1}$$

where $\widehat{A}$ is correct and self-adjoint on $\mathbb{H}$, C is an $m \times m$ matrix with $\operatorname{rank} C = n \leq m$, *and $F_1, \ldots, F_m$ are linearly independent elements of $D(\widehat{A})$. Then*

(i) *B is a self-adjoint operator if and only if C is a Hermitian operator*;

(ii) *B is a correct operator if and only if*

$$\det\left[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C\right] \neq 0; \tag{3.2}$$

(iii) *if B is a correct operator, then* $\dim R(B - \widehat{A}) = n$;

(iv) *the unique solution of* (3.1), *where B is a correct operator, for every $f \in \mathbb{H}$ is given by the formula*

$$x = B^{-1}f = \widehat{A}^{-1}f + FC\left[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C\right]^{-1}\langle f, F^t\rangle_{H^m}. \tag{3.3}$$

Remark 3.2. The correctness of B and the solution (3.3) of (3.1) in Theorem 3.1 do not rely on the linear independence of the components of $F_1, \ldots, F_m$.

This assertion follows immediately from Remark 3.1 of [11] if we suppose that any functional $\Phi_0 \in \mathbb{H}^*$ in the Hilbert space $\mathbb{H}$ can be identified, by Riesz' theorem, with a unique element $F_0 \in \mathbb{H}$ such that

$$\Phi_0(x) = (\Phi_0, x)_H = \langle x, F_0\rangle_H \quad \text{for all} \quad x \in \mathbb{H}, \tag{3.4}$$

so (3.4) easily implies that any vector $\Phi = (\Phi_1, \ldots, \Phi_m) \in \mathbb{H}^{*m}$ can be identified with a unique vector $F = (F_1, \ldots, F_m) \in \mathbb{H}^m$ such that for all $x \in \mathbb{H}$,

$$\begin{aligned} &\Phi_i(x) = (\Phi_i, x)_H = \langle x, F_i\rangle_H, \quad i = 1, \ldots, m, \quad \text{or} \\ &\Phi^t(x) = (\Phi^t, x)_H = \langle x, F^t\rangle_H, \end{aligned} \tag{3.5}$$

and for all vectors $G = (g_1, \ldots, g_m) \in \mathbb{H}^m$,

$$\Phi^t(G) = (\Phi^t, G)_{H^m} = \langle G^t, F\rangle_{H^m}^t = \overline{\langle F^t, G\rangle}_{H^m}. \tag{3.6}$$

Now it is obvious that $\Phi_1, \ldots, \Phi_m$ is a set of linearly independent elements of $\mathbb{H}^{*m}$ if and only if $F_1, \ldots, F_m$ are linearly independent on $\mathbb{H}^m$.

Since $\widehat{A}^4$ is a correct self-adjoint operator and the components of $\mathcal{F}$ are linearly independent, Theorem 3.1 easily implies the following theorem.

Theorem 3.3. *Let $B_1 : \mathbb{H} \to \mathbb{H}$ and*

$$B_1 x = \widehat{A}^4 x - (\widehat{A}^4 \mathcal{F}) \mathbb{C}_{4m} \langle \widehat{A}^4 x, \mathcal{F}^t \rangle_{H^{4m}} = f, \quad D(B_1) = D(\widehat{A}^4), \tag{3.7}$$

where $\widehat{A}$ is as in Theorem 3.1, $\mathbb{C}_{4m}$ is a $(4m) \times (4m)$ matrix with rank $\mathbb{C}_{4m} = n \le 4m$, *and the components of the vector $\mathcal{F} = (\widehat{A}^{-3}F, \widehat{A}^{-2}F, \widehat{A}^{-1}F, F)$ are linearly independent elements of $D(\widehat{A}^4)$. Then*

(i) *B_1 is a self-adjoint operator if and only if $\mathbb{C}_{4m}$ is Hermitian;*

(ii) *B_1 is a correct operator if and only if*

$$\det L_1 = \det \left[I_{4m} - \overline{\langle \widehat{A}^4 \mathcal{F}^t, \mathcal{F} \rangle}_{H^{4m}} \mathbb{C}_{4m} \right] \neq 0; \tag{3.8}$$

(iii) *if B_1 is a correct operator, then* $\dim R(B_1 - \widehat{A}^4) = n$;

(iv) *the unique solution of (3.7), where B_1 is a correct operator, for every $f \in \mathbb{H}$ is given by the formula*

$$x = B_1^{-1} f = \widehat{A}^{-4} f + \mathcal{F} \mathbb{C}_{4m} \left[I_{4m} - \overline{\langle \widehat{A}^4 \mathcal{F}^t, \mathcal{F} \rangle}_{H^{4m}} \mathbb{C}_{4m} \right]^{-1} \langle f, \mathcal{F}^t \rangle_{H^{4m}}. \tag{3.9}$$

Lemma 3.4. *Let $B, B_4 : \mathbb{H} \to \mathbb{H}$ be the operators defined by*

$$Bx = \widehat{A}x - G\langle \widehat{A}x, F^t \rangle_{H^m} = f, \quad D(B) = D(\widehat{A}), \tag{3.10}$$

$$\begin{aligned} B_4 x = {} & \widehat{A}^4 x - V\langle \widehat{A}x, F^t \rangle_{H^m} - Y\langle \widehat{A}^2 x, F^t \rangle_{H^m} - S\langle \widehat{A}^3 x, F^t \rangle_{H^m} - G\langle \widehat{A}^4 x, F^t \rangle_{H^m} = f, \\ & D(B_4) = D(\widehat{A}^4), \end{aligned} \tag{3.11}$$

where $\widehat{A}$ is a linear operator on $\mathbb{H}$, G is a vector of $D(\widehat{A}^3)^m$, the vectors V, Y, S, G satisfy the equations

$$V = \widehat{A}Y - G\overline{\langle F^t, \widehat{A}Y \rangle}_{H^m}, \quad Y = \widehat{A}S - G\overline{\langle F^t, \widehat{A}S \rangle}_{H^m}, \quad S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G \rangle}_{H^m}, \tag{3.12}$$

and the components of the vector $F = (F_1, \ldots, F_m)$ belong to $D(\widehat{A}^4)$. Then $B_4 = B^4$, i.e. B_4 is a biquadratic operator.

Proof. From (3.10) and (3.12) we get

$$\begin{aligned} BG &= \widehat{A}G - G\overline{\langle F^t, \widehat{A}G \rangle}_{H^m} = S, \\ BS &= \widehat{A}S - G\overline{\langle F^t, \widehat{A}S \rangle}_{H^m} = Y, \\ BY &= \widehat{A}Y - G\overline{\langle F^t, \widehat{A}Y \rangle}_{H^m} = V. \end{aligned}$$

Taking this into account and using relation (3.10) for every $x \in D(\widehat{A}^4) \bigcap D(B^4)$ from (3.11), we obtain

$$\begin{aligned} B_4 x &= \widehat{A}^4 x - BY\langle \widehat{A}x, F^t \rangle_{H^m} - BS\langle \widehat{A}^2 x, F^t \rangle_{H^m} - BG\langle \widehat{A}^3 x, F^t \rangle_{H^m} - G\langle \widehat{A}^4 x, F^t \rangle_{H^m} \\ &= B(\widehat{A}^3 x) - BY\langle \widehat{A}x, F^t \rangle_{H^m} - BS\langle \widehat{A}^2 x, F^t \rangle_{H^m} - BG\langle \widehat{A}^3 x, F^t \rangle_{H^m} \\ &= B\big(\widehat{A}^3 x - Y\langle \widehat{A}x, F^t \rangle_{H^m} - S\langle \widehat{A}^2 x, F^t \rangle_{H^m} - G\langle \widehat{A}^3 x, F^t \rangle_{H^m}\big). \end{aligned}$$

In [13, Lemma 3.3] we showed that for the operator B_3 defined by

$$B_3 x = \widehat{A}^3 x - Y\langle \widehat{A}x, F^t \rangle_{H^m} - S\langle \widehat{A}^2 x, F^t \rangle_{H^m} - G\langle \widehat{A}^3 x, F^t \rangle_{H^m} = f, \quad D(B_3) = D(\widehat{A}^3), \tag{3.13}$$

the equations $B_3 = B^3$ and $D(B^3) = D(\widehat{A}^3)$ hold. Thus $B_4 x = B^4 x$ for every $x \in D(\widehat{A}^4) \bigcap D(B^4)$. Now we show that $D(B^4) = D(\widehat{A}^4)$. From $D(B^3) = D(\widehat{A}^3)$ we have $D(B^4) = \{x \in D(\widehat{A}^3) : B^3 x \in D(\widehat{A})\}$. Let $x \in D(\widehat{A}^4)$. Then from (3.13), since $Y, S, G \in D(\widehat{A})$, we see that $x \in D(B^4)$. Now let $x \in D(B^4)$. Again from (3.13), since $Y, S, G \in D(\widehat{A})$, we conclude that $x \in D(\widehat{A}^4)$. Thus $D(B^4) = D(\widehat{A}^4)$ and $B_4 = B^4$. □

We now present the main result of this paper. For a biquadratic operator B_4, we prove a correctness and self-adjointness criterion in terms of the matrices C and give explicit representations for the unique solution of the equation $B_4 x = f$ that are substantially simpler than in the general case of non-biquadratic operators.

Theorem 3.5. *Let $\widehat{A}, B_4 : \mathbb{H} \to \mathbb{H}$ and G, S, Y, V be defined as in Lemma 3.4. We also assume that $\widehat{A}$ is a correct operator, $G = (\widehat{A}F)C$, where C is an $m \times m$ matrix with* rank $C = n \leq m$, *and the components of the vector $\mathcal{F} = (\widehat{A}^{-3}F, \widehat{A}^{-2}F, \widehat{A}^{-1}F, F)$ $\big($respectively, $\widehat{A}^3\mathcal{F} = (F, \widehat{A}F, \widehat{A}^2F, \widehat{A}^3F)\big)$ are linearly independent elements of $D(\widehat{A}^4)$ $\big($respectively, $D(\widehat{A})\big)$. Then*

(i) *B_4 is a self-adjoint operator if and only if C is Hermitian;*

(ii) *B_4 is a correct operator if and only if*

$$\det L = \det\left[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C\right] \neq 0; \tag{3.14}$$

(iii) *if B_4 is a correct operator, then $\dim R(B_4 - \widehat{A}^4) = 4n$ $(n \leq m)$;*

(iv) *the unique solution of the problem (3.11), where B_4 is correct, for every $f \in \mathbb{H}$ is given by the formula*

$$\begin{aligned} x = B_4^{-1}f = \widehat{A}^{-4}f &+ \Big[\widehat{A}^{-3}F + (\widehat{A}^{-2}F)CL^{-1}\overline{\langle F^t, F\rangle}_{H^m} + (\widehat{A}^{-1}F)W + FCL^{-1}\big(\overline{\langle \widehat{A}^{-2}F^t, F\rangle}_{H^m} \\ &+ \overline{\langle \widehat{A}^{-1}F^t, F\rangle}_{H^m} CL^{-1}\overline{\langle F^t, F\rangle}_{H^m} + \overline{\langle F^t, F\rangle}_{H^m} W\big)\Big] CL^{-1} \times \langle f, F^t\rangle_{H^m} \\ &+ \Big[\widehat{A}^{-2}F + (\widehat{A}^{-1}F)CL^{-1}\overline{\langle F^t, F\rangle}_{H^m} + FW\Big] CL^{-1}\langle f, \widehat{A}^{-1}F^t\rangle_{H^m} \\ &+ \Big[\widehat{A}^{-1}F + FCL^{-1}\overline{\langle F^t, F\rangle}_{H^m}\Big] CL^{-1}\langle f, \widehat{A}^{-2}F^t\rangle_{H^m} + FCL^{-1}\langle f, \widehat{A}^{-3}F^t\rangle_{H^m}, \end{aligned} \tag{3.15}$$

where $W = CL^{-1}\left[\overline{\langle \widehat{A}^{-1}F^t, F\rangle}_{H^m} + \overline{\langle F^t, F\rangle}_{H^m} CL^{-1}\overline{\langle F^t, F\rangle}_{H^m}\right]$.

Proof. (i), (iii). Let

$$\begin{aligned} T &= \overline{\langle F^t, F\rangle}_{H^m}, \quad D = \overline{\langle \widehat{A}F^t, F\rangle}_{H^m}, \\ K &= \overline{\langle \widehat{A}^2F^t, F\rangle}_{H^m}, \quad P = \overline{\langle \widehat{A}^3F^t, F\rangle}_{H^m}, \\ M &= \overline{\langle \widehat{A}^4F^t, F\rangle}_{H^m}, \quad H = \overline{\langle \widehat{A}^{-1}F^t, F\rangle}_{H^m}, \\ N &= \overline{\langle \widehat{A}^{-2}F^t, F\rangle}_{H^m}, \quad Z = C(P - KCK)C. \end{aligned}$$

Then the matrix L in (3.14) can be written as $L = I_m - DC$, and the vectors S, Y, V in (3.11) can be written as

$$\begin{aligned} S &= (\widehat{A}^2F)C - (\widehat{A}F)CKC, \\ Y &= (\widehat{A}^3F)C - (\widehat{A}^2F)CKC - (\widehat{A}F)Z, \\ V &= (\widehat{A}^4F)C - (\widehat{A}^3F)CKC - (\widehat{A}^2F)Z - (\widehat{A}F)C(MC - PCKC - KZ). \end{aligned}$$

Equation (3.11) can also be written in matrix notation as

$$\begin{gathered} B_4x = \widehat{A}^4x - (\widehat{A}F, \widehat{A}^2F, \widehat{A}^3F, \widehat{A}^4F) \\ \times \begin{pmatrix} -C(MC - PCKC - KZ) & -Z & -CKC & C \\ -Z & -CKC & C & [0]_m \\ -CKC & C & [0]_m & [0]_m \\ C & [0]_m & [0]_m & [0]_m \end{pmatrix} \begin{pmatrix} \langle \widehat{A}x, F^t\rangle_{H^m} \\ \langle \widehat{A}^2x, F^t\rangle_{H^m} \\ \langle \widehat{A}^3x, F^t\rangle_{H^m} \\ \langle \widehat{A}^4x, F^t\rangle_{H^m} \end{pmatrix} = f, \\ \text{or} \quad B_4x = \widehat{A}^4x - (\widehat{A}^4\mathcal{F})\mathbb{C}_{4m}\langle \widehat{A}^4x, \mathcal{F}^t\rangle_{H^{4m}} = f, \end{gathered} \tag{3.16}$$

where $\mathcal{F} = (\widehat{A}^{-3}F, \widehat{A}^{-2}F, \widehat{A}^{-1}F, F)$,

$$\mathbb{C}_{4m} = \begin{pmatrix} -C(MC - PCKC - KZ) & -Z & -CKC & C \\ -Z & -CKC & C & [0]_m \\ -CKC & C & [0]_m & [0]_m \\ C & [0]_m & [0]_m & [0]_m \end{pmatrix}.$$

It is easy to verify that $\mathbb{C}_{4m}$ is a Hermitian matrix with $\operatorname{rank}\mathbb{C}_{4m} = 4n$ if and only if C is Hermitian with $\operatorname{rank} C = n$. Then, by Theorem 3.3, $\dim R(B_4 - \widehat{A}^4) = 4n$ (where $n \le m$), and the operator B_4 is self-adjoint if and only if C is Hermitian.

(ii) Let $Q = C(MC - PCKC - KZ)$. By Theorem 3.3, the operator B_4 is correct if and only if (3.8) holds true with B_1 replaced by B_4 and L_1 by L_4. We find

$$
\begin{aligned}
L_4 &= I_{4m} - \overline{\langle \widehat{A}^4 \mathcal{F}^t, \mathcal{F}\rangle}_{H^{4m}} \mathbb{C}_{4m} \\
&= I_{4m} - \overline{\begin{pmatrix} \langle F^t, \widehat{A}^{-2}F\rangle_{H^m} & \langle F^t, \widehat{A}^{-1}F\rangle_{H^m} & \langle F^t, F\rangle_{H^m} & \langle \widehat{A}F^t, F\rangle_{H^m} \\ \langle F^t, \widehat{A}^{-1}F\rangle_{H^m} & \langle F^t, F\rangle_{H^m} & \langle \widehat{A}F^t, F\rangle_{H^m} & \langle \widehat{A}^2F^t, F\rangle_{H^m} \\ \langle F^t, F\rangle_{H^m} & \langle \widehat{A}F^t, F\rangle_{H^m} & \langle \widehat{A}^2F^t, F\rangle_{H^m} & \langle \widehat{A}^3F^t, F\rangle_{H^m} \\ \langle \widehat{A}F^t, F\rangle_{H^m} & \langle \widehat{A}^2F^t, F\rangle_{H^m} & \langle \widehat{A}^3F^t, F\rangle_{H^m} & \langle \widehat{A}^4F^t, F\rangle_{H^m} \end{pmatrix}} \mathbb{C}_{4m} \\
&= I_{4m} + \begin{pmatrix} N & H & T & D \\ H & T & D & K \\ T & D & K & P \\ D & K & P & M \end{pmatrix} \begin{pmatrix} Q & Z & CKC & -C \\ Z & CKC & -C & [0]_m \\ CKC & -C & [0]_m & [0]_m \\ -C & [0]_m & [0]_m & [0]_m \end{pmatrix} \\
&= \begin{pmatrix} J_1 & NZ + HCKC - TC & NCKC - HC & -NC \\ J_2 & I_m + HZ + TCKC - DC & HCKC - TC & -HC \\ J_3 & TZ + DCKC - KC & I_m + TCKC - DC & -TC \\ J_4 & DZ + KCKC - PC & DCKC - KC & I_m - DC \end{pmatrix},
\end{aligned} \tag{3.17}
$$

where

$$
\begin{aligned}
J_1 &= I_m + NQ + HZ + TCKC - DC, \\
J_2 &= HQ + TZ + DCKC - KC, \\
J_3 &= TQ + DZ + KCKC - PC, \\
J_4 &= DQ + KZ + PCKC - MC.
\end{aligned}
$$

Multiplying the elements of the second column by KC and adding them to the corresponding elements of the first column, we obtain

$$
\det L_4 = \det \begin{pmatrix} X_1 & NZ + HCKC - TC & NCKC - HC & -NC \\ X_2 & I_m + HZ + TCKC - DC & HCKC - TC & -HC \\ X_3 & TZ + DCKC - KC & I_m + TCKC - DC & -TC \\ X_4 & DZ + KCKC - PC & DCKC - KC & I_m - DC \end{pmatrix},
$$

where

$$
\begin{aligned}
X_1 &= L + NCMC - NCKCPC + HCPC, \\
X_2 &= HCMC - HCKCPC + TCPC, \\
X_3 &= TCMC - TCKCPC - LPC, \\
X_4 &= DCMC - DCKCPC + KCPC - MC.
\end{aligned}
$$

Multiplying the elements of the third column by PC and adding them to the corresponding elements of the first column, we obtain

$$
\det L_4 = \det \begin{pmatrix} L + NCMC & NZ + HCKC - TC & NCKC - HC & -NC \\ HCMC & L + HZ + TCKC & HCKC - TC & -HC \\ TCMC & TZ + DCKC - KC & L + TCKC & -TC \\ -LMC & DZ + KCKC - PC & DCKC - KC & L \end{pmatrix}.
$$

Multiplying the elements of the fourth column by MC and adding them to the corresponding elements of the first column, we obtain

$$
\det L_4 = \det \begin{pmatrix} L & NC(P - KCK)C + HCKC - TC & NCKC - HC & -NC \\ [0]_m & L + HC(P - KCK)C + TCKC & HCKC - TC & -HC \\ [0]_m & TC(P - KCK)C - LKC & L + TCKC & -TC \\ [0]_m & DC(P - KCK)C + KCKC - PC & DCKC - KC & L \end{pmatrix}.
$$

Multiplying the elements of the third column by KC and adding them to the corresponding elements of the second column, we obtain

$$\det L_4 = \det \begin{pmatrix} L & NCPC-TC & NCKC-HC & -NC \\ [0]_m & L+HCPC & HCKC-TC & -HC \\ [0]_m & TCPC & L+TCKC & -TC \\ [0]_m & -LPC & -LKC & L \end{pmatrix}.$$

Multiplying the elements of the fourth column by PC and adding them to the corresponding elements of the second column, we obtain

$$\det L_4 = \det \begin{pmatrix} L & -TC & NCKC-HC & -NC \\ [0]_m & L & HCKC-TC & -HC \\ [0]_m & [0]_m & L+TCKC & -TC \\ [0]_m & [0]_m & -LKC & L \end{pmatrix}.$$

Multiplying the elements of the fourth column by KC and adding them to the corresponding elements of the third column, we obtain

$$\det L_4 = \det \begin{pmatrix} L & -TC & -HC & -NC \\ [0]_m & L & -TC & -HC \\ [0]_m & [0]_m & L & -TC \\ [0]_m & [0]_m & [0]_m & L \end{pmatrix} = (\det L)^4 \neq 0 \Leftrightarrow \det L \neq 0. \tag{3.18}$$

So, by Theorem 3.3 and because of (3.17) and (3.18), the operator B_4 is correct if and only if (3.14) holds true.

(iv) In [13, Theorem 3.4] we showed that

$$\begin{aligned} B_3^{-1} f = \widehat{A}^{-3} f &+ \big[\widehat{A}^{-2} F + (\widehat{A}^{-1} F) C L^{-1} \overline{\langle F^t, F\rangle}_{H^m} \\ &+ FCL^{-1}\big(\overline{\langle \widehat{A}^{-1} F^t, F\rangle}_{H^m} + \overline{\langle F^t, F\rangle}_{H^m} CL^{-1} \overline{\langle F^t, F\rangle}_{H^m}\big)\big] CL^{-1} \langle f, F^t\rangle_{H^m} \\ &+ \big[\widehat{A}^{-1} F + FCL^{-1} \overline{\langle F^t, F\rangle}_{H^m}\big] CL^{-1} \langle f, \widehat{A}^{-1} F^t\rangle_{H^m} + FCL^{-1} \langle f, \widehat{A}^{-2} F^t\rangle_{H^m}. \end{aligned} \tag{3.19}$$

Let $g = B^{-3} f$. Then, in view of (3.3) and (3.19), we have

$$\begin{aligned} B^{-4} f &= B^{-1} g = \widehat{A}^{-1} g + FCL^{-1} \overline{\langle F^t, g\rangle}_{H^m} \\ &= \widehat{A}^{-1}\big\{\widehat{A}^{-3} f + \big[\widehat{A}^{-2} F + (\widehat{A}^{-1} F) CL^{-1} T + FCL^{-1}\big(H + TCL^{-1} T\big)\big] CL^{-1} \langle f, F^t\rangle_{H^m} \\ &\quad + \big(\widehat{A}^{-1} F + FCL^{-1} T\big) CL^{-1} \langle f, \widehat{A}^{-1} F^t\rangle_{H^m} + FCL^{-1} \langle f, \widehat{A}^{-2} F^t\rangle_{H^m}\big\} \\ &\quad + FCL^{-1}\big\{\overline{\langle F^t, \widehat{A}^{-3} f\rangle}_{H^m} + \big(N + HCL^{-1} T + TW\big) CL^{-1} \langle f, F^t\rangle_{H^m} \\ &\quad + (H + TCL^{-1} T) CL^{-1} \langle f, \widehat{A}^{-1} F^t\rangle_{H^m} + TCL^{-1} \langle f, \widehat{A}^{-2} F^t\rangle_{H^m}\big\} \\ &= \widehat{A}^{-4} f + \big[\widehat{A}^{-3} F + (\widehat{A}^{-2} F) CL^{-1} T + (\widehat{A}^{-1} F) W\big] CL^{-1} \langle f, F^t\rangle_{H^m} \\ &\quad + \big[\widehat{A}^{-2} F + (\widehat{A}^{-1} F) CL^{-1} T\big] CL^{-1} \langle f, \widehat{A}^{-1} F^t\rangle_{H^m} \\ &\quad + (\widehat{A}^{-1} F) CL^{-1} \langle f, \widehat{A}^{-2} F^t\rangle_{H^m} + FCL^{-1}\big[\langle f, \widehat{A}^{-3} F^t\rangle_{H^m} \\ &\quad + (N + HCL^{-1} T + TW) CL^{-1} \langle f, F^t\rangle_{H^m} \\ &\quad + (H + TCL^{-1} T) CL^{-1} \langle f, \widehat{A}^{-1} F^t\rangle_{H^m} + TCL^{-1} \langle f, \widehat{A}^{-2} F^t\rangle_{H^m}\big] \\ &= \widehat{A}^{-4} f + \big[\widehat{A}^{-3} F + (\widehat{A}^{-2} F) CL^{-1} T + (\widehat{A}^{-1} F) W \\ &\quad + FCL^{-1}(N + HCL^{-1} T + TW)\big] CL^{-1} \langle f, F^t\rangle_{H^m} \\ &\quad + \big[\widehat{A}^{-2} F + (\widehat{A}^{-1} F) CL^{-1} T + FW\big] CL^{-1} \langle f, \widehat{A}^{-1} F^t\rangle_{H^m} \\ &\quad + (\widehat{A}^{-1} F + FCL^{-1} T) CL^{-1} \langle f, \widehat{A}^{-2} F^t\rangle_{H^m} + FCL^{-1} \langle f, \widehat{A}^{-3} F^t\rangle_{H^m}, \end{aligned}$$

which gives (3.15). The theorem is proved. □

The proof of the previous theorem and Remark 3.2 immediately imply the following observation.

Remark 3.6. The correctness of B_4 and the solution of $B_4x = f$ in Theorem 3.5 do not rely on the linear independence of the components of the vector $\mathcal{F}$.

Remark 3.7. In applications we encounter operators B_1 of the form

$$B_1 u = \widehat{A}^4 u - W_{1m}\langle u, J_1^t\rangle_{H^m} - W_{2m}\langle u, J_2^t\rangle_{H^m} - W_{3m}\langle u, J_3^t\rangle_{H^m} - W_{4m}\langle u, J_4^t\rangle_{H^m} = f,$$
$$D(B_1) = D(\widehat{A}^4), \tag{3.20}$$

where $J_i, W_{im} \in \mathbb{H}^m$, $i = 1,2,3,4$. Then we are interested in knowing whether or not the operator B_1 is a B_4-type operator defined by (3.11) and, therefore, Theorem 3.5 applies. For this purpose, we proceed as follows:

1. We show that the operator $\widehat{A}$ in (3.20) is correct and self-adjoint.
2. We find a vector $F \in D(\widehat{A}^4)^m$ and $m \times m$ matrices M_i, $i = 1,2,3,4$, with constant elements such that

$$\langle u, J_1^t\rangle_{H^m} = M_1\langle \widehat{A}u, F^t\rangle_{H^m},$$
$$\langle u, J_2^t\rangle_{H^m} = M_2\langle \widehat{A}^2u, F^t\rangle_{H^m},$$
$$\langle u, J_3^t\rangle_{H^m} = M_3\langle \widehat{A}^3u, F^t\rangle_{H^m},$$

and

$$\langle u, J_4^t\rangle_{H^m} = M_4\langle \widehat{A}^4u, F^t\rangle_{H^m}.$$

3. We find vectors $V = W_{1m}M_1 \in \mathbb{H}^m$, $Y = \overline{W_{2m}M_2} \in D(\widehat{A})^m$, $S = \overline{W_{3m}M_3} \in D(\widehat{A}^2)^m$, and $G = \overline{W_{4m}M_4} \in D(\widehat{A}^3)^m$ to satisfy the equations $V = \widehat{A}Y - G\overline{\langle F^t, \widehat{A}Y\rangle}_{H^m}$, $Y = \widehat{A}S - G\overline{\langle F^t, \widehat{A}S\rangle}_{H^m}$, and $S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m}$.

If one of these steps fails, then B_1 is not identified as a B_4-type operator and, therefore, the theory cannot be applied.

Below $H^i(0,1)$ denotes the Sobolev space of all complex functions from $L_2(0,1)$ that have generalized derivatives up to the ith order that are Lebesgue integrable, $i = 1,2,3,4$. In the example presented below, we used the programs Derive and Mathematica 6 for computing integrals and some complex expressions. We recall (see [10, p. 780]) that the operator $\widehat{A} : L_2(0,1) \to L_2(0,1)$ defined by

$$\widehat{A}u = iu' = f, \quad D(\widehat{A}) = \{u(t) \in H^1(0,1) : u(0) + u(1) = 0\}, \tag{3.21}$$

is correct and self-adjoint and the unique solution u of the problem (3.21) is given by the formula

$$u(t) = \widehat{A}^{-1}f(t) = \frac{i}{2}\int_0^1 f(x)\,dx - i\int_0^t f(x)\,dx \quad \text{for every } f \in H. \tag{3.22}$$

Then (see [13, p. 424]) the operator $\widehat{A}^2$ defined by

$$\begin{aligned} &\widehat{A}^2u = -u'' = f,\\ &D(\widehat{A}^2) = \{u \in H^2(0,1) : u(0) + u(1) = 0,\ u'(0) + u'(1) = 0\}, \end{aligned} \tag{3.23}$$

is correct and self-adjoint, and for every $f \in L_2(0,1)$ the unique solution u of the problem (3.23) is given by the formula

$$u(t) = \widehat{A}^{-2}f(t) = -\int_0^t (t-x)f(x)\,dx + \frac{1}{4}\int_0^1 (2t - 2x + 1)f(x)\,dx. \tag{3.24}$$

Also (see [13, Proposition 3.6]), the operator $\widehat{A}^3$ defined by

$$\widehat{A}^3u = -iu''' = f, \tag{3.25}$$
$$D(\widehat{A}^3)=\{u \in H^3(0,1) : u(0) + u(1)=0,\ u'(0) + u'(1) = 0,\ u''(0) + u''(1)=0\},$$

is correct and self-adjoint, and for every $f \in L_2(0,1)$ the unique solution u of the problem (3.25) is given by the formula

$$u(t) = \widehat{A}^{-3}f(t) = \frac{i}{2}\int_0^t (t-x)^2 f(x)\,dx - \frac{i}{4}\int_0^1 (t-x)(t-x+1)f(x)\,dx. \tag{3.26}$$

Proposition 3.8. *Let $\widehat{A}$ be the operator defined by (3.21). Then the operator $\widehat{A}^4 : L_2(0,1) \to L_2(0,1)$ defined by*

$$\begin{gathered}\widehat{A}^4 u = u^{(4)} = f, \\ D(\widehat{A}^4) = \{u \in H^4(0,1) \,:\, u^{(k)}(0) + u^{(k)}(1) = 0,\, k = 0,1,2,3\},\end{gathered} \tag{3.27}$$

is correct and self-adjoint, and for every $f \in L_2(0,1)$ the unique solution u of the problem (3.27) is given by the formula

$$u(t) = \widehat{A}^{-4} f(t) = \frac{1}{6}\int_0^t (t-x)^3 f(x)\,dx + \frac{1}{48}\int_0^1 \big[4x^3 - 6x^2(2t+1) + 12tx(t+1) - 4t^3 - 6t^2 + 1\big] f(x)\,dx \tag{3.28}$$

Proof. The correctness and self-adjointness of $\widehat{A}$ imply the correctness and self-adjointness of $\widehat{A}^3$. Now we will prove (3.28). Let $y(x) = \widehat{A}^{-3} f(x)$. Then, by (3.22), (3.26), and Fubini's theorem, we have

$$\begin{aligned}
\widehat{A}^{-4} f(t) &= \widehat{A}^{-1}\big(\widehat{A}^{-3} f(t)\big) = \widehat{A}^{-1} y(t) = \frac{i}{2}\int_0^1 y(z)\,dz - i\int_0^t y(z)\,dz \\
&= \frac{i}{2}\int_0^1 \Big[\frac{i}{2}\int_0^z (z-x)^2 f(x)\,dx - \frac{i}{4}\int_0^1 (z-x)(z-x+1) f(x)\,dx\Big]\,dz \\
&\quad - i\int_0^t \Big[\frac{i}{2}\int_0^z (z-x)^2 f(x)\,dx - \frac{i}{4}\int_0^1 (z-x)(z-x+1) f(x)\,dx\Big]\,dz \\
&= \frac{-1}{4}\int_0^1 dz \int_0^z (z-x)^2 f(x)\,dx + \frac{1}{8}\int_0^1 dz \int_0^1 (z-x)(z-x+1) f(x)\,dx \\
&\quad + \frac{1}{2}\int_0^t dz \int_0^z (z-x)^2 f(x)\,dx - \frac{1}{4}\int_0^t dz \int_0^1 (z-x)(z-x+1) f(x)\,dx \\
&= -\frac{1}{4}\int_0^1 f(x)\,dx \int_x^1 (z-x)^2\,dz + \frac{1}{8}\int_0^1 f(x)\,dx \int_0^1 (z-x)(z-x+1)\,dz \\
&\quad + \frac{1}{2}\int_0^t f(x)\,dx \int_x^t (z-x)^2\,dz - \frac{1}{4}\int_0^1 f(x)\,dx \int_0^t (z-x)(z-x+1)\,dz \\
&= \frac{1}{12}\int_0^1 (x-1)^3 f(x)\,dx + \frac{1}{48}\int_0^1 (6x^2 - 12x + 5) f(x)\,dx + \frac{1}{6}\int_0^t (t-x)^3 f(x)\,dx \\
&\quad - \frac{t}{24}\int_0^1 \big[6x^2 - 6x(t+1) + 2t^2 + 3t\big] f(x)\,dx = \frac{1}{6}\int_0^t (t-x)^3 f(x)\,dx \\
&\quad + \frac{1}{48}\int_0^1 \big[4x^3 - 6x^2(2t+1) + 12tx(t+1) - 4t^3 - 6t^2 + 1\big] f(x)\,dx,
\end{aligned} \tag{3.29}$$

which gives (3.28). □

Example 3.9. The operator $B_1 : L_2(0,1) \to L_2(0,1)$ that corresponds to the problem

$$B_1 u = u^{(4)} + 80\Big\{\frac{620}{21}(t^2 - t) - \frac{65654570}{250047}(t^4 - 2t^3 + t) - i\Big[6t - 3 - \frac{310^2}{63^2}(4t^3 - 6t^2 + 1)\Big]\Big\}$$
$$\times \int_0^1 (2x^5 - 5x^4 + 5x^2 - 1)u'(x)\,dx + 80\Big\{\frac{155}{63}(4t^3 - 6t^2 + 1)$$
$$- i\Big[3t^2 - 3t - \frac{310^2}{63^2}(t^4 - 2t^3 + t)\Big]\Big\} \int_0^1 u''(x)(2x^5 - 5x^4 + 5x^2 - 1)\,dx$$
$$- 20i\Big[\frac{620i}{63}(t^4 - 2t^3 + t) + 4t^3 - 6t^2 + 1\Big] \int_0^1 u'''(x)(2x^5 - 5x^4 + 5x^2 - 1)\,dx$$
$$+ 200i(t^4 - 2t^3 + t) \int_0^1 u'''(x)(x^4 - 2x^3 + x)\,dx = f(t), \quad D(B_1) = D(\widehat{A}^4), \tag{3.30}$$

is correct and self-adjoint, and the unique solution of (3.30) for every $f \in L_2(0,1)$ is given by the formula

$$u(t) = \widehat{A}^{-4} f(t) + \frac{2}{3}\Big[\frac{i}{56}(t^8 - 4t^7 + 14t^5 - 28t^3 + 17t) - \frac{691}{38808}(8t^7 - 28t^6 + 70t^4 - 84t^2 + 17)$$
$$- \frac{691^2 i}{693^2}(t^6 - 3t^5 + 5t^3 - 3t) + \frac{9452636909}{2884375494}(2t^5 - 5t^4 + 5t^2 - 1)\Big] \int_0^1 (2x^5 - 5x^4 + 5x^2 - 1)f(x)\,dx$$
$$+ \frac{2i}{9}\Big[\frac{1}{56}(8t^7 - 28t^6 + 70t^4 - 84t^2 + 17) + \frac{691i}{693}(t^6 - 3t^5 + 5t^3 - 3t) - \frac{691^2}{160083}(2t^5 - 5t^4 + 5t^2 - 1)\Big]$$
$$\times \int_0^1 (x^6 - 3x^5 + 5x^3 - 3x)f(x)\,dx - \frac{1}{252}\Big[-i(t^6 - 3t^5 + 5t^3 - 3t) \tag{3.31}$$
$$+ \frac{691}{231}(2t^5 - 5t^4 + 5t^2 - 1)\Big] \int_0^1 (8x^7 - 28x^6 + 70x^4 - 84x^2 + 17)f(x)\,dx$$
$$+ \frac{i}{84}(2t^5 - 5t^4 + 5t^2 - 1) \int_0^1 (x^8 - 4x^7 + 14x^5 - 28x^3 + 17x)f(x)\,dx,$$

where $\widehat{A}^{-4} f(t)$ is defined by (3.28).

Proof. We apply Theorem 3.5. If we compare (3.30) with (3.11), it is natural to take $\widehat{A}^4 u = u^{(4)}$ with $D(\widehat{A}^4) = D(B_1)$, $m = 1$, $F = 2t^5 - 5t^4 + 5t^2 - 1$. Then we can take $\widehat{A}$ defined by (3.21), $\widehat{A}^2$ defined by (3.23), and $\widehat{A}^3$ defined by (3.25). It is obvious that $F \in D(\widehat{A}^4)$, $\widehat{A}F = 10i(t^4 - 2t^3 + t)$, $\widehat{A}^2 F = -10(4t^3 - 6t^2 + 1)$, $\widehat{A}^3 F = -120i(t^2 - t)$, $\widehat{A}^4 F = 120(2t - 1)$, and that

$$\langle \widehat{A}u, F\rangle_H = \int_0^1 iu'(x)(2t^5 - 5t^4 + 5t^2 - 1)\,dx,$$

$$\langle \widehat{A}^2 u, F\rangle_H = -\int_0^1 u''(x)(2t^5 - 5t^4 + 5t^2 - 1)\,dx,$$

$$\langle \widehat{A}^3 u, F\rangle_H = -i\int_0^1 u'''(x)(2t^5 - 5t^4 + 5t^2 - 1)\,dx,$$

$$\langle \widehat{A}^4 u, F\rangle_H = \int_0^1 u^{(4)}(x)(2x^5 - 5x^4 + 5x^2 - 1)\,dx.$$

Then

$$\int_0^1 u'(x)(2x^5-5x^4+5x^2-1)\,dx = -i\langle \widehat{A}u, F\rangle_H,$$

$$\int_0^1 u''(x)(2x^5-5x^4+5x^2-1)\,dx = -\langle \widehat{A}^2u, F\rangle_H,$$

$$\int_0^1 u'''(x)(2x^5-5x^4+5x^2-1)\,dx = i\langle \widehat{A}^3u, F\rangle_H.$$

Integrating by parts, we obtain

$$\langle \widehat{A}^4u, F\rangle_H = -10\int_0^1 u'''(x)(x^4-2x^3+x)\,dx.$$

Then

$$\int_0^1 u'''(x)(x^4-2x^3+x)\,dx = -\frac{1}{10}\langle \widehat{A}^4u, F\rangle_H.$$

Substituting these formulas into (3.30), we see that

$$\begin{aligned}
B_1u = \widehat{A}^4u &- 80i\Big\{\frac{620}{21}(t^2-t) - \frac{65654570}{250047}(t^4-2t^3+t) - i\Big[6t-3-\frac{310^2}{63^2}(4t^3-6t^2+1)\Big]\Big\}\langle \widehat{A}u, F\rangle_H \\
&- 80\Big\{\frac{155}{63}(4t^3-6t^2+1) - i\Big[3t^2-3t-\frac{310^2}{63^2}(t^4-2t^3+t)\Big]\Big\}\langle \widehat{A}^2u, F\rangle_H \\
&+ 20\Big[\frac{620i}{63}(t^4-2t^3+t)+4t^3-6t^2+1\Big]\langle \widehat{A}^3u, F\rangle_H \\
&- 20i(t^4-2t^3+t)\langle \widehat{A}^4u, F\rangle_H = f(t).
\end{aligned} \tag{3.32}$$

Again comparing (3.32) with (3.11), we obtain

$$\begin{aligned}
V &= 80i\Big\{\frac{620}{21}(t^2-t) - \frac{65654570}{250047}(t^4-2t^3+t) - i\Big[6t-3-\frac{310^2}{63^2}(4t^3-6t^2+1)\Big]\Big\}, \\
Y &= 80\Big\{\frac{155}{63}(4t^3-6t^2+1) - i\Big[3t^2-3t-\frac{310^2}{63^2}(t^4-2t^3+t)\Big]\Big\}, \\
S &= -20\Big[\frac{620i}{63}(t^4-2t^3+t)+4t^3-6t^2+1\Big], \quad \text{and} \quad G = 20i(t^4-2t^3+t).
\end{aligned}$$

It is obvious that $G \in D(\widehat{A}^3)$. The vectors $F, \widehat{A}F, \widehat{A}^2F, \widehat{A}^3F$ are linearly independent elements of $D(\widehat{A})$, since the corresponding determinant of the Gram matrix is nonzero. Using Derive, we obtain

$$\begin{aligned}
\widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m} &= -20(4t^3-6t^2+1) - 20i(t^4-2t^3+t)\frac{620}{63} = S, \\
\widehat{A}S - G\overline{\langle F^t, \widehat{A}S\rangle}_{H^m} &= -20i\Big[\frac{620i}{63}(4t^3-6t^2+1)+12(t^2-t)\Big] - 20i(t^4-2t^3+t)\Big(-\frac{620^2}{63^2}\Big) = Y,
\end{aligned}$$

and

$$\widehat{A}Y - G\overline{\langle F^t, \widehat{A}Y\rangle}_{H^m} = 80i\Big\{\frac{155}{63}(12t^2-12t) - i\Big[6t-3-\frac{310^2}{63^2}(4t^3-6t^2+1)\Big]\Big\} - 20i(t^4-2t^3+t)\frac{262618280}{250047} = V.$$

The last three equalities, by Lemma 3.4, show that the operator B_1 is biquadratic, i.e., $B_1 = B_4$. From $G = (\widehat{A}F)C$ it follows that $20i(t^4-2t^3+t) = 10i(t^4-2t^3+t)C$. This equation implies that $C = 2$. Using the program Derive, we find $\langle F^t, F\rangle_H = \frac{691}{1386}$ $\langle \widehat{A}F^t, F\rangle_H = 0$. By Theorem 3.5, the operator B_1 is correct and self-adjoint, since $C = 2$ is a real number and

$$\det L = \det[I_m - \overline{\langle \widehat{A}F^t, F\rangle}_{H^m} C] = 1 - 0 = 1 \neq 0.$$

Then $L^{-1} = 1$. If in (3.22), (3.24), and (3.26) we substitute $f = F = 2t^5 - 5t^4 + 5t^2 - 1$, we obtain

$$\widehat{A}^{-1}F = -\frac{i}{3}(t^6 - 3t^5 + 5t^3 - 3t),$$

$$\widehat{A}^{-2}F = -\frac{1}{168}(8t^7 - 28t^6 + 70t^4 - 84t^2 + 17),$$

and

$$\widehat{A}^{-3}F = \frac{i}{168}(t^8 - 4t^7 + 14t^5 - 28t^3 + 17t).$$

Then

$$\langle f, \widehat{A}^{-1}F\rangle_H = -\frac{i}{3}\int_0^1 (x^6 - 3x^5 + 5x^3 - 3x) f(x)\,dx,$$

$$\langle f, \widehat{A}^{-2}F\rangle_H = -\frac{1}{168}\int_0^1 (8x^7 - 28x^6 + 70x^4 - 84x^2 + 17) f(x)\,dx,$$

and

$$\langle f, \widehat{A}^{-3}F\rangle_H = \frac{i}{168}\int_0^1 (x^8 - 4x^7 + 14x^5 - 28x^3 + 17x) f(x)\,dx.$$

Using the program Derive, we obtain $\langle \widehat{A}^{-1}F, F\rangle_H = 0$, $\langle \widehat{A}^{-2}F, F\rangle_H = \frac{5461}{108108}$, and it follows from (3.15) that $W = \frac{4\cdot 691^2}{1386^2}$. As a result of this and (3.15), we obtain the solution (3.31) of the problem (3.30). □

A comment from the first and third authors. The results of this paper were proved together with our friend P. C. Tsekrekos who passed away from a heart attack in the fall of 2009, at the age of 64. We would like to express our deepest sorrow for his sudden death.

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